

Unit C2

Vector spaces

Introduction

In this unit you will meet a mathematical structure that is one of the most important unifying concepts of pure mathematics. It is that of a *vector space*. A vector space consists of a set of elements called *vectors*, and two operations: *addition of vectors* and *multiplication by a scalar*. These *vectors* need not be vectors in the geometric sense given in Book A; instead, they may be a wide range of objects including complex numbers, functions and matrices.

You will first consider properties of \mathbb{R}^2 and \mathbb{R}^3 , and see how these two- and three-dimensional spaces lead not only to n -dimensional space \mathbb{R}^n , but also to the formal definition of a vector space. You will meet a variety of quite different vector spaces and study various concepts relating to vector spaces. For example, you will meet the idea of a *subspace* of a vector space, which is a subset of a vector space that is itself a vector space; this is similar to the relationship between subgroups and groups, which you met in Book B.

The theory of vector spaces introduced in this unit will underpin the remaining units of this book.

1 Vector spaces

In Book A you met the plane and three-dimensional space. In this section you will see that properties that you are familiar with in these two- and three-dimensional spaces also hold for other, quite different-looking *spaces*.

1.1 Euclidean spaces

Recall from Unit A1 *Sets, functions and vectors* that \mathbb{R}^2 is the set of all ordered pairs of real numbers, and \mathbb{R}^3 is the set of all ordered triples of real numbers. You saw that we can interpret these sets as the plane and as three-dimensional space, respectively, in the following two ways. We can interpret their elements first as the coordinates of points with respect to a specified coordinate system, and second as vectors in component form with respect to this coordinate system.

In this way, once axes have been specified, we can consider the elements of \mathbb{R}^2 equivalently as ordered pairs, as points in the plane, or as vectors in the plane. And likewise for \mathbb{R}^3 , we can consider the elements equivalently as ordered triples of real numbers, as points in three-dimensional space or as vectors in three-dimensional space.

Also in Unit A1, you met two operations: addition of vectors and multiplication of a vector by a scalar. These operations are defined on \mathbb{R}^2 and \mathbb{R}^3 as follows.

Definitions

In \mathbb{R}^2 , the set of ordered pairs of real numbers, the operations of **addition** and of **multiplication by a scalar** are defined as:

$$(u_1, u_2) + (v_1, v_2) = (u_1 + v_1, u_2 + v_2),$$

$$\alpha(u_1, u_2) = (\alpha u_1, \alpha u_2), \quad \text{where } \alpha \in \mathbb{R}.$$

In \mathbb{R}^3 , the set of ordered triples of real numbers, the operations of **addition** and of **multiplication by a scalar** are defined as:

$$(u_1, u_2, u_3) + (v_1, v_2, v_3) = (u_1 + v_1, u_2 + v_2, u_3 + v_3),$$

$$\alpha(u_1, u_2, u_3) = (\alpha u_1, \alpha u_2, \alpha u_3), \quad \text{where } \alpha \in \mathbb{R}.$$

It turns out that \mathbb{R}^2 and \mathbb{R}^3 are particular instances of a class of mathematical structures called *vector spaces*. In this unit you will meet many other examples, and study the properties that are common to all of them.

You are familiar with vectors in \mathbb{R}^2 and \mathbb{R}^3 , but there is no reason to stop at \mathbb{R}^3 – why not consider \mathbb{R}^4 , \mathbb{R}^5 , or even \mathbb{R}^n , for larger positive integers n ?

Definitions

Let n be a positive integer. An **ordered n -tuple** is a sequence of real numbers (u_1, u_2, \dots, u_n) . The set of all ordered n -tuples is called **n -dimensional space**, and is denoted by \mathbb{R}^n .

To highlight the connection between n -dimensional space (for a positive integer n), denoted by \mathbb{R}^n , and 2- and 3-dimensional space with geometrical vectors, the space \mathbb{R}^n is often called a **Euclidean space** and its elements (u_1, u_2, \dots, u_n) are called vectors. For example, \mathbb{R}^4 is the four-dimensional Euclidean space of vectors with four components.

Although it is difficult to visualise vectors in spaces with dimension greater than three, it is possible to carry out exactly the same algebraic manipulations with these vectors, and it turns out that these spaces are also *vector spaces*.

Vector addition and scalar multiplication in \mathbb{R}^n are defined as in \mathbb{R}^2 and \mathbb{R}^3 .

Definitions

Let

$$\mathbf{u} = (u_1, u_2, \dots, u_n) \text{ and } \mathbf{v} = (v_1, v_2, \dots, v_n)$$

be two vectors in \mathbb{R}^n . The operations of **addition** and of **multiplication by a scalar** are defined as:

$$\begin{aligned}\mathbf{u} + \mathbf{v} &= (u_1, u_2, \dots, u_n) + (v_1, v_2, \dots, v_n) \\ &= (u_1 + v_1, u_2 + v_2, \dots, u_n + v_n), \\ \alpha \mathbf{u} &= (\alpha u_1, \alpha u_2, \dots, \alpha u_n), \quad \text{where } \alpha \in \mathbb{R}.\end{aligned}$$

Worked Exercise C20

Let $\mathbf{u} = (1, 1, \dots, 1)$ and $\mathbf{v} = (1, 2, \dots, n)$ be two vectors in \mathbb{R}^n . Form the vectors $\mathbf{u} + \mathbf{v}$ and $2\mathbf{u}$.

Solution

$$\mathbf{u} + \mathbf{v} = (1, 1, \dots, 1) + (1, 2, \dots, n) = (2, 3, \dots, n+1)$$

$$2\mathbf{u} = 2(1, 1, \dots, 1) = (2, 2, \dots, 2)$$

Exercise C44

Let $\mathbf{u} = (1, -1, 2, 0, -3)$ and $\mathbf{v} = (0, 2, -1, 4, 0)$ be two vectors in \mathbb{R}^5 . Form the vectors $\mathbf{u} + \mathbf{v}$ and $-3\mathbf{u}$.

This method of generalisation (here from \mathbb{R}^2 and \mathbb{R}^3 to \mathbb{R}^n) is common throughout mathematics. We start with spaces like \mathbb{R}^2 and \mathbb{R}^3 that we can visualise and look at their properties, and then we generalise these properties to spaces that we cannot easily visualise, such as \mathbb{R}^n . So we go from particular cases to a general case.

We can go even further, and think of a vector with a never-ending list of components (v_1, v_2, v_3, \dots) . This is hard to visualise, but is not difficult to handle mathematically. The set of such vectors is called \mathbb{R}^∞ , and is an infinite-dimensional vector space. (You will meet a formal definition of *dimension* of a vector space in Section 3.) Vector addition and scalar multiplication are again performed component-wise.

Worked Exercise C21

Let $\mathbf{u} = (1, 0, 1, 0, 1, \dots)$ and $\mathbf{v} = (1, -2, 3, -4, 5, \dots)$ be two vectors in \mathbb{R}^∞ . Form the vectors $\mathbf{u} + \mathbf{v}$ and $5\mathbf{u}$.

Solution

$$\begin{aligned}\mathbf{u} + \mathbf{v} &= (1, 0, 1, 0, 1, \dots) + (1, -2, 3, -4, 5, \dots) \\ &= (2, -2, 4, -4, 6, \dots)\end{aligned}$$

$$5\mathbf{u} = 5(1, 0, 1, 0, 1, \dots) = (5, 0, 5, 0, 5, \dots)$$

1.2 Real vector spaces

Before meeting the definition of a vector space, we will look at \mathbb{R}^4 and a set of polynomials, and will observe that, despite their apparent differences, these sets share many important properties.

The space \mathbb{R}^4

A vector in \mathbb{R}^4 has the form (v_1, v_2, v_3, v_4) , where v_1, v_2, v_3 and v_4 are real numbers, and the operations of vector addition and scalar multiplication are as defined in the previous subsection.

If we have two vectors $\mathbf{u} = (u_1, u_2, u_3, u_4)$ and $\mathbf{v} = (v_1, v_2, v_3, v_4)$ in \mathbb{R}^4 , then their sum is

$$\begin{aligned}\mathbf{u} + \mathbf{v} &= (u_1, u_2, u_3, u_4) + (v_1, v_2, v_3, v_4) \\ &= (u_1 + v_1, u_2 + v_2, u_3 + v_3, u_4 + v_4).\end{aligned}$$

This last vector also belongs to \mathbb{R}^4 because each of the four components is a real number, so \mathbb{R}^4 is *closed under vector addition*; that is, the closure property (A1), which you met in Unit A2 *Number systems*, holds for the addition of vectors in \mathbb{R}^4 .

For example, if $\mathbf{u} = (1, 3, 5, 7)$ and $\mathbf{v} = (2, -1, -5, 6)$ are vectors in \mathbb{R}^4 , then

$$\mathbf{u} + \mathbf{v} = (1, 3, 5, 7) + (2, -1, -5, 6) = (3, 2, 0, 13),$$

which is a vector in \mathbb{R}^4 .

In fact addition of vectors in \mathbb{R}^4 satisfies all the usual rules of arithmetic, as follows. The next worked exercise proves the commutative property (A5) and the additive identity property (A3), and you are asked to prove the remaining two properties in the following exercise.

Addition of vectors in \mathbb{R}^4 **A1 Closure** For all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^4$,

$$\mathbf{u} + \mathbf{v} \in \mathbb{R}^4.$$

A2 Associativity For all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^4$,

$$(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w}).$$

A3 Additive identity For all $\mathbf{v} \in \mathbb{R}^4$, and $\mathbf{0} \in \mathbb{R}^4$,

$$\mathbf{v} + \mathbf{0} = \mathbf{v} = \mathbf{0} + \mathbf{v}.$$

A4 Additive inverses For each $\mathbf{v} \in \mathbb{R}^4$, there is a vector $-\mathbf{v} \in \mathbb{R}^4$ such that

$$\mathbf{v} + (-\mathbf{v}) = \mathbf{0} = -\mathbf{v} + \mathbf{v}.$$

A5 Commutativity For all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^4$,

$$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}.$$

Worked Exercise C22Prove that the following properties hold for vector addition in \mathbb{R}^4 .

- (a) The commutative property (A5): $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.
- (b) The additive identity property (A3): $\mathbf{v} + \mathbf{0} = \mathbf{v} = \mathbf{0} + \mathbf{v}$, where $\mathbf{0}$ is the zero vector $(0, 0, 0, 0)$.

SolutionLet $\mathbf{u} = (u_1, u_2, u_3, u_4)$ and $\mathbf{v} = (v_1, v_2, v_3, v_4)$.

$$\begin{aligned}
 \text{(a) } \mathbf{u} + \mathbf{v} &= (u_1, u_2, u_3, u_4) + (v_1, v_2, v_3, v_4) \\
 &= (u_1 + v_1, u_2 + v_2, u_3 + v_3, u_4 + v_4) \\
 &= (v_1 + u_1, v_2 + u_2, v_3 + u_3, v_4 + u_4) \\
 &= (v_1, v_2, v_3, v_4) + (u_1, u_2, u_3, u_4) \\
 &= \mathbf{v} + \mathbf{u}
 \end{aligned}$$

Therefore the commutative property (A5) holds.

$$\begin{aligned}
 \text{(b) } \mathbf{v} + \mathbf{0} &= (v_1, v_2, v_3, v_4) + (0, 0, 0, 0) \\
 &= (v_1 + 0, v_2 + 0, v_3 + 0, v_4 + 0) \\
 &= (v_1, v_2, v_3, v_4) = \mathbf{v}
 \end{aligned}$$

Also, using the commutative property (A5) from part (a) we have

$$\mathbf{v} + \mathbf{0} = \mathbf{v} = \mathbf{0} + \mathbf{v},$$

so the additive identity property (A3) holds.

Exercise C45

Prove that the following properties hold for vector addition in \mathbb{R}^4 .

- (a) The associative property (A2): $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$.
- (b) The additive inverses property (A4): $\mathbf{v} + (-\mathbf{v}) = \mathbf{0} = -\mathbf{v} + \mathbf{v}$, where $\mathbf{v} = (v_1, v_2, v_3, v_4)$ and $-\mathbf{v} = (-v_1, -v_2, -v_3, -v_4)$.

Recall from Unit B1 *Symmetry* that a set with a binary operation is a **group** if the following four axioms hold:

G1 (closure); G2 (associativity); G3 (identity) and G4 (inverses).

The first four properties (A1–A4) of vector addition in \mathbb{R}^4 show that the set \mathbb{R}^4 under the operation of vector addition satisfies these four properties; that is, $(\mathbb{R}^4, +)$ is a group with additive identity the zero vector $(0, 0, 0, 0)$, and $-\mathbf{v}$ the additive inverse of \mathbf{v} . The final property, commutativity (A5), shows that it is in fact an abelian group.

These properties all involve vector addition, but \mathbb{R}^4 also has some properties that involve scalar multiplication.

Let $\mathbf{v} = (v_1, v_2, v_3, v_4) \in \mathbb{R}^4$ and $\alpha \in \mathbb{R}$. Then

$$\alpha\mathbf{v} = \alpha(v_1, v_2, v_3, v_4) = (\alpha v_1, \alpha v_2, \alpha v_3, \alpha v_4).$$

This vector also belongs to \mathbb{R}^4 , so \mathbb{R}^4 is *closed under scalar multiplication*.

For example, if $\mathbf{v} = (1, 2, -5, -3) \in \mathbb{R}^4$ and $\alpha = 4$, then

$$\alpha\mathbf{v} = 4(1, 2, -5, -3) = (4, 8, -20, -12),$$

which belongs to \mathbb{R}^4 .

Note that if you multiply a vector in \mathbb{R}^4 by $\beta \in \mathbb{R}$, and then by $\alpha \in \mathbb{R}$, you obtain the same result as multiplying by $\alpha\beta$. This is because, for all $\alpha, \beta \in \mathbb{R}$ and $\mathbf{v} = (v_1, v_2, v_3, v_4) \in \mathbb{R}^4$,

$$\begin{aligned} \alpha(\beta\mathbf{v}) &= \alpha(\beta(v_1, v_2, v_3, v_4)) \\ &= \alpha(\beta v_1, \beta v_2, \beta v_3, \beta v_4) \\ &= (\alpha\beta v_1, \alpha\beta v_2, \alpha\beta v_3, \alpha\beta v_4) \\ &= (\alpha\beta)(v_1, v_2, v_3, v_4) \\ &= (\alpha\beta)\mathbf{v}. \end{aligned}$$

For example, if $\mathbf{v} = (1, 2, -5, -3) \in \mathbb{R}^4$ and $\alpha = 4$, $\beta = -2$, then

$$\begin{aligned} \alpha(\beta\mathbf{v}) &= 4(-2(1, 2, -5, -3)) \\ &= 4(-2, -4, 10, 6) \\ &= (-8, -16, 40, 24) \\ &= (-8)(1, 2, -5, -3) \\ &= (\alpha\beta)\mathbf{v}. \end{aligned}$$

Also, if $\mathbf{v} = (v_1, v_2, v_3, v_4)$, then

$$1\mathbf{v} = 1(v_1, v_2, v_3, v_4) = (v_1, v_2, v_3, v_4) = \mathbf{v}.$$

These properties of scalar multiplication of vectors in \mathbb{R}^4 can be summarised as follows.

Scalar multiplication of vectors in \mathbb{R}^4

S1 Closure For all $\mathbf{v} \in \mathbb{R}^4$, and $\alpha \in \mathbb{R}$,

$$\alpha \mathbf{v} \in \mathbb{R}^4.$$

S2 Associativity For all $\mathbf{v} \in \mathbb{R}^4$, and $\alpha, \beta \in \mathbb{R}$,

$$\alpha(\beta \mathbf{v}) = (\alpha\beta) \mathbf{v}.$$

S3 Scalar multiplicative identity For all $\mathbf{v} \in \mathbb{R}^4$,

$$1\mathbf{v} = \mathbf{v}.$$

Finally, there are two distributive properties that connect vector addition and scalar multiplication.

For example, if $\mathbf{u} = (1, 3, 5, 7)$ and $\mathbf{v} = (2, -1, -5, 6)$ are vectors in \mathbb{R}^4 , and $\alpha = 3$ and $\beta = 4$, then

$$\begin{aligned} \alpha(\mathbf{u} + \mathbf{v}) &= 3((1, 3, 5, 7) + (2, -1, -5, 6)) \\ &= 3(3, 2, 0, 13) \\ &= (9, 6, 0, 39) \end{aligned}$$

and

$$\begin{aligned} \alpha \mathbf{u} + \alpha \mathbf{v} &= 3(1, 3, 5, 7) + 3(2, -1, -5, 6) \\ &= (3, 9, 15, 21) + (6, -3, -15, 18) \\ &= (9, 6, 0, 39), \end{aligned}$$

which illustrates the first distributive property. Also,

$$\begin{aligned} (\alpha + \beta) \mathbf{v} &= (3 + 4)(2, -1, -5, 6) \\ &= 7(2, -1, -5, 6) \\ &= (14, -7, -35, 42) \end{aligned}$$

and

$$\begin{aligned} \alpha \mathbf{v} + \beta \mathbf{v} &= 3(2, -1, -5, 6) + 4(2, -1, -5, 6) \\ &= (6, -3, -15, 18) + (8, -4, -20, 24) \\ &= (14, -7, -35, 42), \end{aligned}$$

which illustrates the second.

These properties connecting vector addition and scalar multiplication can be summarised as follows.

Combining addition and scalar multiplication of vectors in \mathbb{R}^4

D1 Distributivity For all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^4$, and $\alpha \in \mathbb{R}$,

$$\alpha(\mathbf{u} + \mathbf{v}) = \alpha\mathbf{u} + \alpha\mathbf{v}.$$

D2 Distributivity For all $\mathbf{v} \in \mathbb{R}^4$, and $\alpha, \beta \in \mathbb{R}$,

$$(\alpha + \beta)\mathbf{v} = \alpha\mathbf{v} + \beta\mathbf{v}.$$

The space of quadratic polynomials

Let us now look at another, apparently very different set of elements. This is the set of *quadratic polynomials*, namely, functions of the form

$$\begin{aligned} p : \mathbb{R} &\longrightarrow \mathbb{R} \\ x &\longmapsto a + bx + cx^2, \end{aligned}$$

where $a, b, c \in \mathbb{R}$. We call this set P_3 because it comprises all the real polynomials of degree less than 3. Thus

$$P_3 = \{p(x) : p(x) = a + bx + cx^2, a, b, c \in \mathbb{R}\}.$$

Here we have used the convention from Book A that when a real function is specified only by a rule, it is understood that the domain of the function is the set of all real numbers for which the rule is applicable, and the codomain of the function is \mathbb{R} .

(We write the terms of the polynomial in increasing order of powers here, as usually done when working within a *vector space* of polynomials.)

To simplify the notation further, we write

$$P_3 = \{a + bx + cx^2 : a, b, c \in \mathbb{R}\}.$$

This set includes the quadratic polynomials (where c is non-zero), the linear polynomials (where c is 0 and b is non-zero) and constants (where b and c are 0 and a is non-zero), as well as the zero polynomial (where $a = b = c = 0$). At first sight, there is no reason why this set of elements should have the properties that we have just shown are satisfied by \mathbb{R}^4 ; however, these properties all hold for this set as well.

First we consider the properties A1–A5 involving addition.

Consider $p_1(x) = a_1 + b_1x + c_1x^2$ and $p_2(x) = a_2 + b_2x + c_2x^2$, then

$$\begin{aligned} p_1(x) + p_2(x) &= (a_1 + b_1x + c_1x^2) + (a_2 + b_2x + c_2x^2) \\ &= (a_1 + a_2) + (b_1 + b_2)x + (c_1 + c_2)x^2, \end{aligned}$$

which also belongs to P_3 . Therefore the closure property (A1) holds for addition in P_3 .

For example, $3 + 4x - 2x^2$ and $5 - 3x + 7x^2$ both belong to P_3 , and

$$(3 + 4x - 2x^2) + (5 - 3x + 7x^2) = 8 + x + 5x^2,$$

which also belongs to P_3 . The next worked exercise proves the commutative property (A5) and the additive inverses property (A4), and you are asked to prove the remaining two properties in the following exercise.

Worked Exercise C23

Prove that the following properties hold for addition in P_3 .

- (a) The commutative property (A5): $p_1(x) + p_2(x) = p_2(x) + p_1(x)$.
- (b) The additive inverses property (A4):
 $p_1(x) + (-p_1(x)) = \mathbf{0} = -p_1(x) + p_1(x)$.

Solution

$$\begin{aligned} \text{(a)} \quad p_1(x) + p_2(x) &= (a_1 + b_1x + c_1x^2) + (a_2 + b_2x + c_2x^2) \\ &= (a_1 + a_2) + (b_1 + b_2)x + (c_1 + c_2)x^2 \\ &= (a_2 + a_1) + (b_2 + b_1)x + (c_2 + c_1)x^2 \\ &= (a_2 + b_2x + c_2x^2) + (a_1 + b_1x + c_1x^2) \\ &= p_2(x) + p_1(x) \end{aligned}$$

Therefore the commutative property (A5) holds for addition in P_3 .

$$\begin{aligned} \text{(b)} \quad p_1(x) + (-p_1(x)) &= (a_1 + b_1x + c_1x^2) + (-a_1 - b_1x - c_1x^2) \\ &= (a_1 - a_1) + (b_1 - b_1)x + (c_1 - c_1)x^2 \\ &= 0 + 0x + 0x^2 = \mathbf{0} \end{aligned}$$

Also, using the commutative property (A5) from part (a) we have

$$p_1(x) + (-p_1(x)) = \mathbf{0} = -p_1(x) + p_1(x),$$

so the additive inverses property (A4) holds for addition in P_3 .

Exercise C46

Prove that the following properties hold for addition in P_3 .

- (a) The associative property (A2):
 $(p_1(x) + p_2(x)) + p_3(x) = p_1(x) + (p_2(x) + p_3(x))$.
- (b) The additive identity property (A3): $p_1(x) + \mathbf{0} = p_1(x) = \mathbf{0} + p_1(x)$,
 where $\mathbf{0} = 0 + 0x + 0x^2$ is the zero polynomial in P_3 .

It follows that P_3 satisfies the same addition properties as \mathbb{R}^4 , and therefore P_3 is also an abelian group under addition.

We can multiply a polynomial through by a real constant; that is, by a scalar. In fact P_3 has the same properties involving scalar multiplication as \mathbb{R}^4 .

Let $p(x) = a + bx + cx^2$ and $\alpha \in \mathbb{R}$, then

$$\alpha p(x) = \alpha(a + bx + cx^2) = (\alpha a) + (\alpha b)x + (\alpha c)x^2,$$

which also belongs to P_3 . So P_3 is *closed under scalar multiplication*; that is, the closure property (S1) holds for P_3 under scalar multiplication.

In the following exercise you are asked to check the remaining properties involving scalar multiplication (S2 and S3), for a particular case.

Exercise C47

Let $p(x) = 1 - x + 2x^2$ and $\alpha = 2$, $\beta = -3$. Show that the following properties hold for these scalars and this quadratic polynomial.

- (a) The identity property (S3): $1 \times p(x) = p(x)$.
- (b) The associative property (S2): $\alpha(\beta p(x)) = (\alpha\beta)p(x)$.

To finish looking at the properties of P_3 , we note that the distributive properties (D1 and D2) that connect addition and scalar multiplication hold for P_3 ; the proofs simply involve multiplying out brackets. For all $p_1(x), p_2(x) \in P_3$ and $\alpha, \beta \in \mathbb{R}$,

$$\alpha(p_1(x) + p_2(x)) = \alpha p_1(x) + \alpha p_2(x)$$

and

$$(\alpha + \beta)p_1(x) = \alpha p_1(x) + \beta p_1(x).$$

So \mathbb{R}^4 and P_3 satisfy the same set of properties with respect to addition and scalar multiplication, even though \mathbb{R}^4 is a Euclidean space and P_3 is a set of polynomials. The idea that connects them is the concept of a *vector space*.

Vector space definition

In Book B we studied symmetries of geometric figures, and then abstracted the properties to obtain the definition of a group. We go through a similar process here. We have just studied \mathbb{R}^4 and P_3 , and we now abstract from them the definition of a *vector space*. We then go on to look at other examples of vector spaces. The elements of these vector spaces are of diverse types: complex numbers, functions, matrices, and many others.

The definition of a vector space is one of the longest definitions in mathematics. It looks formidable, but the axioms A1–A5, S1–S3 and D1–D2 are precisely the properties we checked for \mathbb{R}^4 and P_3 . Thus this

definition follows naturally from our previous examples. As for \mathbb{R}^4 and P_3 , axioms A1–A5 refer to vector addition (implying that a vector space is an abelian group under addition), S1–S3 refer to scalar multiplication, and D1–D2 to how we combine these operations. Therefore a *vector space* is a set of objects called *vectors* that can be added together and scalar multiplied in such a way that all the usual properties of arithmetic hold. Thus the definition includes the properties for addition, the properties for scalar multiplication and the properties of how these two operations combine.

Definition

A **real vector space** consists of a set V of elements called **vectors** and two operations, vector addition and scalar multiplication, such that the following axioms hold.

Axioms for addition

A1 Closure For all $\mathbf{v}_1, \mathbf{v}_2 \in V$,

$$\mathbf{v}_1 + \mathbf{v}_2 \in V.$$

A2 Associativity For all $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in V$,

$$(\mathbf{v}_1 + \mathbf{v}_2) + \mathbf{v}_3 = \mathbf{v}_1 + (\mathbf{v}_2 + \mathbf{v}_3).$$

A3 Additive identity For all $\mathbf{v} \in V$, there is a zero element $\mathbf{0} \in V$ satisfying

$$\mathbf{v} + \mathbf{0} = \mathbf{v} = \mathbf{0} + \mathbf{v}.$$

A4 Additive inverses For each $\mathbf{v} \in V$, there is an element $-\mathbf{v}$ (its additive inverse) such that

$$\mathbf{v} + (-\mathbf{v}) = \mathbf{0} = -\mathbf{v} + \mathbf{v}.$$

A5 Commutativity For all $\mathbf{v}_1, \mathbf{v}_2 \in V$,

$$\mathbf{v}_1 + \mathbf{v}_2 = \mathbf{v}_2 + \mathbf{v}_1.$$

Axioms A1–A5 imply that $(V, +)$ is an *abelian group*.

Axioms for scalar multiplication

S1 Closure For all $\mathbf{v} \in V$, and $\alpha \in \mathbb{R}$,

$$\alpha \mathbf{v} \in V.$$

S2 Associativity For all $\mathbf{v} \in V$, and $\alpha, \beta \in \mathbb{R}$,

$$\alpha(\beta \mathbf{v}) = (\alpha\beta)\mathbf{v}.$$

S3 Scalar multiplicative identity For all $\mathbf{v} \in V$,

$$1\mathbf{v} = \mathbf{v}.$$

Axioms combining addition and scalar multiplication**D1 Distributivity** For all $\mathbf{v}_1, \mathbf{v}_2 \in V$, and $\alpha \in \mathbb{R}$,

$$\alpha(\mathbf{v}_1 + \mathbf{v}_2) = \alpha\mathbf{v}_1 + \alpha\mathbf{v}_2.$$

D2 Distributivity For all $\mathbf{v} \in V$, and $\alpha, \beta \in \mathbb{R}$,

$$(\alpha + \beta)\mathbf{v} = \alpha\mathbf{v} + \beta\mathbf{v}.$$

The word ‘real’ in this definition refers to the fact that *the scalars used in forming scalar multiples are real numbers*; that is, a real vector space is a vector space over the field \mathbb{R} (which means that the scalars are elements in \mathbb{R}). More generally, it is possible to define a vector space over *any* field, so it is also possible to form complex and rational vector spaces, where the vectors are scalar multiplied by complex and rational numbers, respectively. This is because the sets of complex and rational numbers are also fields. However, we are only concerned with real vector spaces in this module.

It is worth noting that \mathbb{R} itself is a real vector space: the fact that the vector space axioms hold for $V = \mathbb{R}$ follows from the field properties that hold for \mathbb{R} , which were shown in Unit A2 when considering the arithmetic of real numbers.

Where we use the term *vector* for the elements of vector spaces, many mathematical texts use the terms *element* and *vector* interchangeably.

Checking the axioms

We now look at the set $V = \{a \cos x + b \sin x : a, b \in \mathbb{R}\}$ of functions, and show that it is a real vector space by checking all the axioms in the definition. You will not be asked to check *all* these axioms in a single exercise: this example simply illustrates how it can be done.

Addition and scalar multiplication are defined on V as follows.

If $a_1 \cos x + b_1 \sin x$ and $a_2 \cos x + b_2 \sin x$ are vectors of V , and $\alpha \in \mathbb{R}$, then

$$\begin{aligned} (a_1 \cos x + b_1 \sin x) + (a_2 \cos x + b_2 \sin x) \\ = (a_1 + a_2) \cos x + (b_1 + b_2) \sin x \end{aligned}$$

and

$$\alpha(a_1 \cos x + b_1 \sin x) = \alpha a_1 \cos x + \alpha b_1 \sin x.$$

For example,

$$(3 \cos x + 2 \sin x) + (4 \cos x - 6 \sin x) = 7 \cos x - 4 \sin x$$

and

$$-5(3 \cos x + 4 \sin x) = -15 \cos x - 20 \sin x.$$

We check the axioms one by one.

A1 Closure V is closed under addition of functions, since, if $a_1 \cos x + b_1 \sin x$ and $a_2 \cos x + b_2 \sin x$ are vectors of V , then

$$\begin{aligned} (a_1 \cos x + b_1 \sin x) + (a_2 \cos x + b_2 \sin x) \\ = (a_1 + a_2) \cos x + (b_1 + b_2) \sin x, \end{aligned}$$

which is a vector of V .

A2 Associativity Addition is associative, since, if $a_1 \cos x + b_1 \sin x$, $a_2 \cos x + b_2 \sin x$ and $a_3 \cos x + b_3 \sin x$ are vectors of V , then

$$\begin{aligned} ((a_1 \cos x + b_1 \sin x) + (a_2 \cos x + b_2 \sin x)) + (a_3 \cos x + b_3 \sin x) \\ = ((a_1 + a_2) \cos x + (b_1 + b_2) \sin x) + (a_3 \cos x + b_3 \sin x) \\ = (a_1 + a_2 + a_3) \cos x + (b_1 + b_2 + b_3) \sin x \end{aligned}$$

and

$$\begin{aligned} (a_1 \cos x + b_1 \sin x) + ((a_2 \cos x + b_2 \sin x) + (a_3 \cos x + b_3 \sin x)) \\ = (a_1 \cos x + b_1 \sin x) + ((a_2 + a_3) \cos x + (b_2 + b_3) \sin x) \\ = (a_1 + a_2 + a_3) \cos x + (b_1 + b_2 + b_3) \sin x. \end{aligned}$$

A3 Additive identity The zero vector is $0 \cos x + 0 \sin x$, since this is in V and, if $a \cos x + b \sin x \in V$, then

$$(a \cos x + b \sin x) + (0 \cos x + 0 \sin x) = a \cos x + b \sin x$$

and

$$(0 \cos x + 0 \sin x) + (a \cos x + b \sin x) = a \cos x + b \sin x.$$

A4 Additive inverses The additive inverse of $a \cos x + b \sin x$ is $-a \cos x - b \sin x$, since this is in V and, if $a \cos x + b \sin x \in V$, then

$$(a \cos x + b \sin x) + (-a \cos x - b \sin x) = 0 \cos x + 0 \sin x$$

and

$$(-a \cos x - b \sin x) + (a \cos x + b \sin x) = 0 \cos x + 0 \sin x.$$

A5 Commutativity Addition is commutative, since, if $a_1 \cos x + b_1 \sin x$ and $a_2 \cos x + b_2 \sin x$ are vectors of V , then

$$\begin{aligned} (a_1 \cos x + b_1 \sin x) + (a_2 \cos x + b_2 \sin x) \\ = (a_1 + a_2) \cos x + (b_1 + b_2) \sin x \end{aligned}$$

and

$$\begin{aligned} (a_2 \cos x + b_2 \sin x) + (a_1 \cos x + b_1 \sin x) \\ = (a_2 + a_1) \cos x + (b_2 + b_1) \sin x \\ = (a_1 + a_2) \cos x + (b_1 + b_2) \sin x. \end{aligned}$$

S1 Closure V is closed under scalar multiplication, since, for $a \cos x + b \sin x \in V$ and $\alpha \in \mathbb{R}$, we have

$$\alpha(a \cos x + b \sin x) = \alpha a \cos x + \alpha b \sin x.$$

This is in V , since $\alpha a, \alpha b \in \mathbb{R}$.

S2 Associativity For $\alpha, \beta \in \mathbb{R}$ and $a \cos x + b \sin x \in V$, we have

$$\begin{aligned}\alpha(\beta(a \cos x + b \sin x)) &= \alpha(\beta a \cos x + \beta b \sin x) \\ &= \alpha\beta a \cos x + \alpha\beta b \sin x \\ &= (\alpha\beta)(a \cos x + b \sin x).\end{aligned}$$

S3 Scalar multiplicative identity For $a \cos x + b \sin x \in V$, we have

$$1(a \cos x + b \sin x) = a \cos x + b \sin x.$$

D1 Distributivity For $\alpha \in \mathbb{R}$ and $a_1 \cos x + b_1 \sin x$ and $a_2 \cos x + b_2 \sin x$ in V , we have

$$\begin{aligned}\alpha((a_1 \cos x + b_1 \sin x) + (a_2 \cos x + b_2 \sin x)) \\ &= \alpha((a_1 + a_2) \cos x + (b_1 + b_2) \sin x) \\ &= \alpha(a_1 + a_2) \cos x + \alpha(b_1 + b_2) \sin x\end{aligned}$$

and

$$\begin{aligned}\alpha(a_1 \cos x + b_1 \sin x) + \alpha(a_2 \cos x + b_2 \sin x) \\ &= \alpha a_1 \cos x + \alpha b_1 \sin x + \alpha a_2 \cos x + \alpha b_2 \sin x \\ &= \alpha(a_1 + a_2) \cos x + \alpha(b_1 + b_2) \sin x.\end{aligned}$$

D2 Distributivity For $\alpha, \beta \in \mathbb{R}$ and $a \cos x + b \sin x \in V$, we have

$$\begin{aligned}(\alpha + \beta)(a \cos x + b \sin x) \\ &= (\alpha + \beta)a \cos x + (\alpha + \beta)b \sin x \\ &= \alpha a \cos x + \alpha b \sin x + \beta a \cos x + \beta b \sin x\end{aligned}$$

and

$$\begin{aligned}\alpha(a \cos x + b \sin x) + \beta(a \cos x + b \sin x) \\ &= \alpha a \cos x + \alpha b \sin x + \beta a \cos x + \beta b \sin x.\end{aligned}$$

Since all the vector space properties are satisfied, V is a vector space.

We now look briefly at some further examples of vector spaces, to give you some idea of the different areas of mathematics in which this concept arises.

The set of linear polynomials P_2

The set P_2 of linear polynomials comprises the real polynomials of degree less than 2; that is, the polynomials of the form $p(x) = a + bx$, where $a, b \in \mathbb{R}$. Vector addition and scalar multiplication are defined on P_2 as follows.

If $p(x) = a + bx$ and $q(x) = c + dx$, and $\alpha \in \mathbb{R}$, then

$$p(x) + q(x) = (a + bx) + (c + dx) = (a + c) + (b + d)x$$

and

$$\alpha p(x) = \alpha(a + bx) = (\alpha a) + (\alpha b)x.$$

The result of each of these operations is a linear polynomial, so P_2 is closed under the operations of addition and scalar multiplication, and therefore satisfies the closure axioms (A1 and S1). The other axioms can be checked in the same way.

More generally, for each positive integer n , the set P_n of real polynomials of degree less than n , with the usual operations of addition and scalar multiplication, is a vector space.

The set of complex numbers \mathbb{C}

The set \mathbb{C} comprises the numbers of the form $a + bi$, where $i^2 = -1$ and $a, b \in \mathbb{R}$. Vector addition and scalar multiplication are defined on \mathbb{C} as

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$

and

$$\alpha(a + bi) = (\alpha a) + (\alpha b)i.$$

This is a *real* vector space because we multiply the *complex* number (the vector) by a *real* number (the scalar).

The result of each of these operations is a complex number, so \mathbb{C} is closed under the operations of vector addition and scalar multiplication, and therefore satisfies the closure axioms (A1 and S1). The other axioms can be checked in the same way.

The set $M_{2,3}$ of 2×3 matrices with real entries

The set $M_{2,3}$ comprises the 2×3 matrices of the form

$$\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}, \quad \text{where } a, b, c, d, e, f \in \mathbb{R}.$$

Vector addition and scalar multiplication are defined on $M_{2,3}$ as follows.

If $\mathbf{A} = \begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} b_1 & b_2 & b_3 \\ b_4 & b_5 & b_6 \end{pmatrix}$, and $\alpha \in \mathbb{R}$, then

$$\begin{aligned} \mathbf{A} + \mathbf{B} &= \begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \end{pmatrix} + \begin{pmatrix} b_1 & b_2 & b_3 \\ b_4 & b_5 & b_6 \end{pmatrix} \\ &= \begin{pmatrix} a_1 + b_1 & a_2 + b_2 & a_3 + b_3 \\ a_4 + b_4 & a_5 + b_5 & a_6 + b_6 \end{pmatrix} \end{aligned}$$

and

$$\alpha \mathbf{A} = \alpha \begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \end{pmatrix} = \begin{pmatrix} \alpha a_1 & \alpha a_2 & \alpha a_3 \\ \alpha a_4 & \alpha a_5 & \alpha a_6 \end{pmatrix}.$$

The result of each of these operations is a 2×3 matrix with real entries, so this set is closed under the operations of vector addition and scalar multiplication, and therefore satisfies the closure axioms (A1 and S1). The other axioms can be checked in the same way.

More generally, for positive integers m and n , the set $M_{m,n}$ of $m \times n$ matrices with real entries is a vector space under the operations of vector addition and scalar multiplication.

The set \mathbb{R}^∞

If $\mathbf{u} = (u_1, u_2, \dots)$ and $\mathbf{v} = (v_1, v_2, \dots)$ belong to \mathbb{R}^∞ , and $\alpha \in \mathbb{R}$, then

$$\mathbf{u} + \mathbf{v} = (u_1, u_2, \dots) + (v_1, v_2, \dots) = (u_1 + v_1, u_2 + v_2, \dots)$$

and

$$\alpha \mathbf{u} = \alpha(u_1, u_2, \dots) = (\alpha u_1, \alpha u_2, \dots).$$

The result of each of these operations is a vector of \mathbb{R}^∞ , so \mathbb{R}^∞ is closed under the operations of vector addition and scalar multiplication, and therefore satisfies the closure axioms (A1 and S1). The other axioms can be checked in the same way.

These examples are only a few of the many real vector spaces. You will meet more of them as you work through this unit, and as you encounter other mathematical concepts in the remainder of this module.



We finish this section by looking at some sets that are not vector spaces. In each case you should assume the usual definitions of addition and scalar multiplication for the elements of these sets to show that these sets are not vector spaces.

Worked Exercise C24

Show that neither of the following sets is a real vector space.



- (a) $V = \{\text{all polynomials of degree equal to } 5\}$
- (b) $V = \{a + bi \in \mathbb{C} : a \geq 0\}$

Solution

- (a)  Recall that P_3 is the set of all polynomials of degree less than 3. Here we have all polynomials of degree *equal* to 5. Therefore the closure axiom (A1) is a good place to start. All that is needed is one pair of polynomials in V whose sum is not in V . 

Consider the polynomials $p(x) = x + x^5$, $q(x) = x - x^5$, both of degree 5. We have $p(x) + q(x) = (x + x^5) + (x - x^5) = 2x$, which is a polynomial of degree 1 and so not in V .

Therefore addition on the set of all polynomials of degree equal to 5 fails to satisfy the closure axiom (A1), so V is not a real vector space.

 Other axioms also fail, or do not make sense; for example, V fails the additive identity axiom (A3) since it contains no zero vector, and therefore the additive inverses property (A4) makes no sense here. 

- (b) 🧠 We know that the set of *all* complex numbers is a vector space, so the condition $a \geq 0$ is important here. The closure axiom (S1) is a good place to start. Can we find a complex number in V where scalar multiplication by, say, $\alpha = -1$ is not in V ? 🧠

Consider $z = 1$ in V , and let $\alpha = -1$, then $\alpha z = -1$, which is not of the form $a + bi$ where $a \geq 0$.

Therefore scalar multiplication on the set of complex numbers of the form $a + bi$, where $a \geq 0$, fails to satisfy the closure axiom (S1), so V is not a real vector space.

🧠 Other axioms also fail; for example, V fails the additive inverses axiom (A4) since $z = 1$ has no additive inverse in V . In addition, because axiom S1 fails, the axioms S2, D1 and D2 are meaningless. 🧠

Exercise C48

Show that neither of the following sets is a real vector space.

- (a) $V = \{(x, y) \in \mathbb{R}^2 : y = 2x + 1\}$
 (b) $V = \left\{ \begin{pmatrix} 0 & a \\ b & c \end{pmatrix} : a, b, c \in \mathbb{Z} \right\}$

2 Linear combinations and spanning sets

In this section you will see that in a vector space, some sets of vectors are special. These special sets are such that every other vector in the space can be produced by adding combinations and scalar multiples of vectors just in this special set.

2.1 Linear combinations

We begin by looking at the different ways in which we can express a single vector in \mathbb{R}^2 as a combination of two other vectors.

For example, the vector $(5, 3)$ in \mathbb{R}^2 , illustrated in Figure 1, can be written as

$$(5, 3) = 5(1, 0) + 3(0, 1).$$

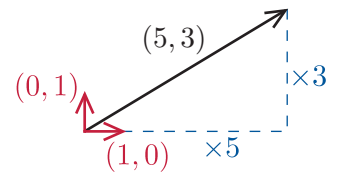


Figure 1 The vector $(5, 3)$ as a linear combination of $(1, 0)$ and $(0, 1)$

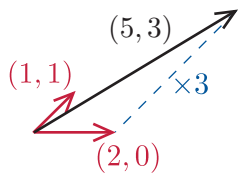


Figure 2 The vector $(5, 3)$ as a linear combination of $(2, 0)$ and $(1, 1)$

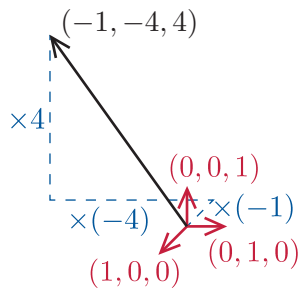


Figure 3 A vector in \mathbb{R}^3 as a linear combination of three vectors

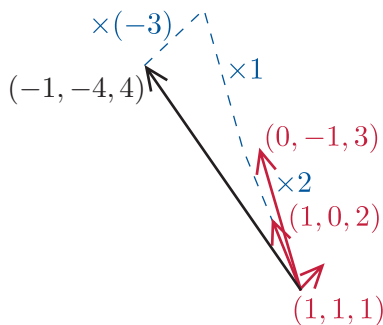


Figure 4 A vector in \mathbb{R}^3 as a linear combination of three vectors

We could also write $(5, 3)$ in terms of $(2, 0)$ and $(1, 1)$, illustrated in Figure 2. In this case we have

$$(5, 3) = 1(2, 0) + 3(1, 1).$$

If you look at the right-hand sides of these equations, you will see that they both have the same form. In each case we have written

$$(5, 3) = \alpha \mathbf{v}_1 + \beta \mathbf{v}_2,$$

where $\mathbf{v}_1 = (1, 0)$, $\mathbf{v}_2 = (0, 1)$, $\alpha = 5$ and $\beta = 3$ in the first case, and $\mathbf{v}_1 = (2, 0)$, $\mathbf{v}_2 = (1, 1)$, $\alpha = 1$ and $\beta = 3$ in the second case.

We call $\alpha \mathbf{v}_1 + \beta \mathbf{v}_2$ a *linear combination* of the two vectors \mathbf{v}_1 and \mathbf{v}_2 .

Because \mathbf{v}_1 and \mathbf{v}_2 are vectors in \mathbb{R}^2 , so are $\alpha \mathbf{v}_1$ and $\beta \mathbf{v}_2$, since they are scalar multiples of \mathbf{v}_1 and \mathbf{v}_2 ; and hence so is $\alpha \mathbf{v}_1 + \beta \mathbf{v}_2$, since it is the sum of two vectors in \mathbb{R}^2 . So $\alpha \mathbf{v}_1 + \beta \mathbf{v}_2$ is also a vector in \mathbb{R}^2 .

Similarly in \mathbb{R}^3 , the vector $(-1, -4, 4)$, illustrated in Figure 3, can be written as

$$(-1, -4, 4) = -1(1, 0, 0) - 4(0, 1, 0) + 4(0, 0, 1)$$

or as illustrated in Figure 4, in terms of the three vectors $(1, 0, 2)$, $(0, -1, 3)$ and $(1, 1, 1)$ as

$$(-1, -4, 4) = 2(1, 0, 2) + 1(0, -1, 3) - 3(1, 1, 1).$$

These are two examples: they are not the only possibilities. Each of these equations has the form

$$(-1, -4, 4) = \alpha \mathbf{v}_1 + \beta \mathbf{v}_2 + \gamma \mathbf{v}_3,$$

where the expression on the right-hand side of the equation is a *linear combination* of three vectors.

These linear combinations of vectors in \mathbb{R}^2 and \mathbb{R}^3 are particular examples of the following definition.

Definition

Let $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ belong to a vector space V . Then a **linear combination** of the vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ is a vector of the form

$$\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \cdots + \alpha_k \mathbf{v}_k,$$

where $\alpha_1, \alpha_2, \dots, \alpha_k$ are real numbers. This vector also belongs to V .

We begin by looking at how we can form linear combinations of vectors, and then investigate whether we can write a particular vector as a linear combination of other vectors in the same vector space.

In the worked exercises and exercises of this section we have tried to keep the arithmetic simple by using *integer* scalar multiples and coordinates. In general, any real numbers may occur.

Worked Exercise C25

- (a) In \mathbb{R}^3 , calculate the linear combination $2\mathbf{v}_1 + 3\mathbf{v}_2$ when $\mathbf{v}_1 = (1, 0, 3)$ and $\mathbf{v}_2 = (0, 2, -1)$.
- (b) In \mathbb{R}^4 , calculate the linear combination $2\mathbf{v}_1 + 3\mathbf{v}_2 + 4\mathbf{v}_3 - \mathbf{v}_4$ when $\mathbf{v}_1 = (1, 0, 3, 1)$, $\mathbf{v}_2 = (0, 2, 0, -1)$, $\mathbf{v}_3 = (0, 1, -2, 0)$ and $\mathbf{v}_4 = (2, 10, -2, -1)$.

Solution

- (a) $2\mathbf{v}_1 + 3\mathbf{v}_2 = 2(1, 0, 3) + 3(0, 2, -1)$
 $= (2, 0, 6) + (0, 6, -3) = (2, 6, 3)$
- (b) $2\mathbf{v}_1 + 3\mathbf{v}_2 + 4\mathbf{v}_3 - \mathbf{v}_4$
 $= 2(1, 0, 3, 1) + 3(0, 2, 0, -1) + 4(0, 1, -2, 0) - (2, 10, -2, -1)$
 $= (2, 0, 6, 2) + (0, 6, 0, -3) + (0, 4, -8, 0) - (2, 10, -2, -1)$
 $= (0, 0, 0, 0)$

Exercise C49

- (a) In \mathbb{R}^2 , let $\mathbf{v}_1 = (0, 3)$ and $\mathbf{v}_2 = (2, 1)$. Calculate the linear combination $4\mathbf{v}_1 - 2\mathbf{v}_2$.
- (b) In \mathbb{R}^4 , let $\mathbf{v}_1 = (1, 2, 1, 3)$ and $\mathbf{v}_2 = (2, 1, 0, -1)$. Calculate the linear combination $3\mathbf{v}_1 + 2\mathbf{v}_2$.

We now look at linear combinations of vectors in vector spaces other than \mathbb{R}^2 , \mathbb{R}^3 and \mathbb{R}^4 . In the worked exercise and exercise that follow, we assume that the operations of vector addition and scalar multiplication for polynomials, matrices and functions are the usual ones.

Worked Exercise C26

For each of the following vector spaces V and vectors $\mathbf{v}_1, \mathbf{v}_2$ and \mathbf{v}_3 in V , form the linear combination $3\mathbf{v}_1 - 2\mathbf{v}_2 + \mathbf{v}_3$.

- (a) $V = P_3$, $\mathbf{v}_1 = 1 + x + x^2$, $\mathbf{v}_2 = 1 - x$, $\mathbf{v}_3 = x + x^2$.
- (b) $V = M_{2,3}$, $\mathbf{v}_1 = \begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & 3 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 2 & -1 & 0 \\ 0 & 3 & -4 \end{pmatrix}$,
 $\mathbf{v}_3 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 1 \end{pmatrix}$.

Solution

$$\begin{aligned} \text{(a)} \quad 3\mathbf{v}_1 - 2\mathbf{v}_2 + \mathbf{v}_3 &= 3(1 + x + x^2) - 2(1 - x) + (x + x^2) \\ &= 1 + 6x + 4x^2 \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad 3\mathbf{v}_1 - 2\mathbf{v}_2 + \mathbf{v}_3 &= 3 \begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & 3 \end{pmatrix} - 2 \begin{pmatrix} 2 & -1 & 0 \\ 0 & 3 & -4 \end{pmatrix} + \begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 1 \end{pmatrix} \\ &= \begin{pmatrix} -2 & 2 & 6 \\ 0 & -7 & 18 \end{pmatrix} \end{aligned}$$

Exercise C50

For each of the following vector spaces V and vectors \mathbf{v}_1 and \mathbf{v}_2 in V , form the linear combination $2\mathbf{v}_1 - 4\mathbf{v}_2$.

- (a) $V = P_3$, $\mathbf{v}_1 = 2 - x + 3x^2$, $\mathbf{v}_2 = -1 + x$.
 (b) V is the set of all real functions, $\mathbf{v}_1 = \sin x$, $\mathbf{v}_2 = x \cos x$.
 (c) $V = M_{2,2}$, $\mathbf{v}_1 = \begin{pmatrix} -1 & 1 \\ 2 & 0 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 3 & 1 \\ 0 & -2 \end{pmatrix}$.

Now that we have formed linear combinations of different numbers of vectors in various vector spaces, we consider the harder problem of deciding whether we can express a given vector as a linear combination of a particular set of vectors. In the next worked exercise, we look at an example before giving a general strategy.

Worked Exercise C27

Determine whether $(3, -1)$ can be expressed as a linear combination of each of the following.

- (a) $\mathbf{v}_1 = (2, 0)$ and $\mathbf{v}_2 = (1, 1)$. (b) $\mathbf{v}_1 = (2, 2)$ and $\mathbf{v}_2 = (1, 1)$.
 (c) $\mathbf{v}_1 = (9, -3)$ and $\mathbf{v}_2 = (-6, 2)$.



Solution

- (a) We need to find real numbers α and β such that

$$(3, -1) = \alpha(2, 0) + \beta(1, 1),$$

that is,

$$(3, -1) = (2\alpha + \beta, \beta).$$

 We equate the two first coordinates (components) to get $3 = 2\alpha + \beta$, and then the two second coordinates (components) to get $-1 = \beta$. 

Equating corresponding coordinates, we obtain the system of linear equations

$$\begin{aligned} 2\alpha + \beta &= 3 \\ \beta &= -1. \end{aligned}$$

Substituting $\beta = -1$ in the first equation gives $\alpha = 2$. So

$$\begin{aligned} (3, -1) &= 2(2, 0) - 1(1, 1) \\ &= 2\mathbf{v}_1 - \mathbf{v}_2. \end{aligned}$$

- (b) We need to find real numbers α and β such that



$$(3, -1) = \alpha(2, 2) + \beta(1, 1),$$

that is,



$$(3, -1) = (2\alpha + \beta, 2\alpha + \beta).$$

Equating corresponding coordinates, we obtain the system

$$\begin{aligned} 2\alpha + \beta &= 3 \\ 2\alpha + \beta &= -1. \end{aligned}$$

 The left-hand sides of these equations are the same but the right-hand sides are different, so we can immediately conclude that they are inconsistent. Alternatively, subtracting the second equation from the first yields the equation $0 = 4$. 

This pair of equations is inconsistent, since no values of α and β satisfy both of them.

 We might have expected this since any linear combination of $(1, 1)$ and $(2, 2)$ must have both coordinates the same. 

We cannot express $(3, -1)$ as a linear combination of these two vectors.

- (c) We need to find real numbers α and β such that



$$(3, -1) = \alpha(9, -3) + \beta(-6, 2),$$

that is,

$$(3, -1) = (9\alpha - 6\beta, -3\alpha + 2\beta).$$

Equating corresponding coordinates, we obtain the system

$$\begin{aligned} 9\alpha - 6\beta &= 3 \\ -3\alpha + 2\beta &= -1. \end{aligned}$$



 Multiplying the second equation by -3 yields the first equation. 

These equations are equivalent to the single equation

$$3\alpha - 2\beta = 1,$$

so any values of α and β that satisfy this equation give a solution. Thus in this case there are infinitely many solutions. For example, if $\alpha = 1$, then $\beta = 1$, and

$$(3, -1) = (9, -3) + (-6, 2) = \mathbf{v}_1 + \mathbf{v}_2.$$

 Other solutions include $\alpha = \frac{1}{3}$, $\beta = 0$ and $\alpha = 0$, $\beta = -\frac{1}{2}$. Here there are infinitely many solutions because both $(9, -3)$ and $(-6, 2)$ are multiples of $(3, -1)$. 

The following strategy describes the method we have just used.

Strategy C6

To determine whether a given vector \mathbf{v} can be written as a linear combination of the vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$:

1. write $\mathbf{v} = \alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \dots + \alpha_k \mathbf{v}_k$
2. use this expression to write down a system of linear equations in the unknowns $\alpha_1, \alpha_2, \dots, \alpha_k$
3. solve the resulting system of equations, if possible.

Then \mathbf{v} can be written as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ if and only if the system has a solution.

Recall from Unit C1 *Linear equations and matrices* that a system of linear equations may have no solution, a unique solution, or infinitely many solutions. Therefore this strategy may give no solution, a unique solution, or infinitely many solutions, as we saw in Worked Exercise C27.

When dealing with polynomial functions, such as those in P_3 , we use the fact that two polynomial equations in the variable x are equal if and only if the coefficients of corresponding powers of x are equal, and *equate corresponding coefficients*.

Worked Exercise C28

- (a) In \mathbb{R}^3 , express the vector $(1, 1, 1)$ as a linear combination of the vectors $(1, 0, 1)$, $(0, 1, 2)$ and $(-1, 1, 0)$.
- (b) In P_3 , express the polynomial $2 + 2x + 5x^2$ as a linear combination of the polynomials $1 + 3x^2$ and $2x - x^2$.

Solution

We follow the steps of Strategy C6.

- (a) Let α , β and γ be real numbers such that

$$(1, 1, 1) = \alpha(1, 0, 1) + \beta(0, 1, 2) + \gamma(-1, 1, 0).$$

Then

$$(1, 1, 1) = (\alpha - \gamma, \beta + \gamma, \alpha + 2\beta).$$

Equating corresponding coordinates, we obtain the system

$$\begin{array}{rcl} \alpha & - \gamma & = 1 \\ & \beta + \gamma & = 1 \\ \alpha + 2\beta & & = 1. \end{array}$$

Adding the first two equations gives $\alpha + \beta = 2$, and solving this and the last equation gives $\beta = -1$ and $\alpha = 3$. Substitution then gives $\gamma = 2$, so the required linear combination is

$$(1, 1, 1) = 3(1, 0, 1) - 1(0, 1, 2) + 2(-1, 1, 0).$$

(You may have used Gauss–Jordan elimination to solve the system of linear equations, rather than solving them directly. Either method is fine.)

- (b) Let α and β be real numbers such that



$$2 + 2x + 5x^2 = \alpha(1 + 3x^2) + \beta(2x - x^2).$$

Then

$$2 + 2x + 5x^2 = \alpha + (2\beta)x + (3\alpha - \beta)x^2.$$

Equating corresponding coefficients, we obtain the system

$$\begin{aligned}\alpha &= 2 \\ 2\beta &= 2 \\ 3\alpha - \beta &= 5.\end{aligned}$$

 The solutions can be read off from the first two equations, but it is important to check that *all* the equations are satisfied: otherwise there is no solution. 

The first two equations have the solution $\alpha = 2$, $\beta = 1$, and this solution also satisfies the third equation. So the required linear combination is

$$2 + 2x + 5x^2 = 2(1 + 3x^2) + (2x - x^2).$$

Exercise C51

- (a) In \mathbb{R}^2 , express the vector $(2, 4)$ as a linear combination of the vectors $(0, 3)$ and $(2, 1)$.
- (b) In \mathbb{R}^3 , express the vector $(2, 3, -2)$ as a linear combination of the vectors $(0, 1, 0)$, $(1, 2, -1)$ and $(1, 1, -2)$.
- (c) In $M_{2,2}$, express the matrix $\begin{pmatrix} 3 & 1 \\ 0 & 4 \end{pmatrix}$ as a linear combination of the matrices $\begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}$ and $\begin{pmatrix} 0 & -2 \\ 0 & 1 \end{pmatrix}$.

2.2 Spanning sets

We now look at the set of vectors that is produced when we form *all possible* linear combinations of a given set of vectors.

Picture any two vectors in \mathbb{R}^2 , and suppose that we form all possible linear combinations of these two vectors. What vectors do we obtain? Are there any vectors in \mathbb{R}^2 that *cannot* be written as a linear combination of these two vectors? (We saw such an example in Worked Exercise C27(b).) What happens if we start with one vector in \mathbb{R}^2 ? If we form all possible linear combinations of it, what vectors can result? What happens if we start with one, two or three vectors in \mathbb{R}^3 ?

Let us start with a set consisting of exactly one vector in \mathbb{R}^2 – namely, the set containing the vector $(1, 0)$. The set of all linear combinations of $(1, 0)$, illustrated in Figure 5, is

$$\{\alpha(1, 0) : \alpha \in \mathbb{R}\} = \{(\alpha, 0) : \alpha \in \mathbb{R}\}.$$

Geometrically, the members of this set are the points on the x -axis in \mathbb{R}^2 . So this set of linear combinations is a line (the x -axis) in \mathbb{R}^2 . We say that the set $\{(1, 0)\}$ *spans* the x -axis, and that the x -axis is *spanned* by $\{(1, 0)\}$.

Suppose that we now take the set $\{(1, 0), (0, 1)\}$ containing two vectors. The set of all linear combinations of $(1, 0)$ and $(0, 1)$, illustrated in Figure 6, is

$$\{\alpha(1, 0) + \beta(0, 1) : \alpha, \beta \in \mathbb{R}\} = \{(\alpha, \beta) : \alpha, \beta \in \mathbb{R}\}.$$

Since α and β can take any real values, this set consists of all the points in \mathbb{R}^2 . We say that $\{(1, 0), (0, 1)\}$ *spans* \mathbb{R}^2 , and that \mathbb{R}^2 is *spanned* by $\{(1, 0), (0, 1)\}$.

We now write down the formal definitions of span and spanning, before looking at some more examples.

Definitions

Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ be a finite set of vectors in a vector space V . Then the **span** $\langle S \rangle$ of S is the set of all possible linear combinations

$$\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \dots + \alpha_k \mathbf{v}_k,$$

where $\alpha_1, \alpha_2, \dots, \alpha_k$ are real numbers; that is,

$$\langle S \rangle = \{\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \dots + \alpha_k \mathbf{v}_k : \alpha_1, \alpha_2, \dots, \alpha_k \in \mathbb{R}\}.$$

We say that the set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ **spans** $\langle S \rangle$ or is a **spanning set** for $\langle S \rangle$, and that $\langle S \rangle$ is the set **spanned** by S .

While S is a *finite* set of vectors, the span $\langle S \rangle$ is generally an infinite set of vectors (such as a line or plane): this is because the linear combinations involve the set of real numbers. In fact, the span $\langle S \rangle$ is itself a vector space, as you will see later, in Subsection 4.1 (Theorem C28).

To test whether a vector \mathbf{v} lies in the span of a given set S , we use Strategy C6 to determine whether \mathbf{v} can be written as a linear combination of the vectors in S .



Figure 5 The linear combinations of $(1, 0)$

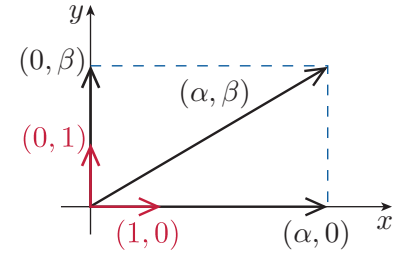


Figure 6 The linear combinations of $(1, 0)$ and $(0, 1)$

Worked Exercise C29

Let $S = \{(1, 1, 0), (0, 1, 1)\}$. Which of the following vectors belong to $\langle S \rangle$?

- (a) $(0, 0, 1)$ (b) $(4, 2, -2)$

Solution



We apply Strategy C6.

- (a) We write

$$(0, 0, 1) = \alpha(1, 1, 0) + \beta(0, 1, 1) = (\alpha, \alpha + \beta, \beta).$$

Equating corresponding coordinates, we obtain the system

$$\begin{aligned}\alpha &= 0 \\ \alpha + \beta &= 0 \\ \beta &= 1.\end{aligned}$$

 Subtracting the first and third equations from the second yields the equation $0 = -1$. 

This system is inconsistent and therefore has no solution. So $(0, 0, 1)$ does not belong to $\langle S \rangle$.

- (b) We write

$$(4, 2, -2) = \alpha(1, 1, 0) + \beta(0, 1, 1) = (\alpha, \alpha + \beta, \beta).$$

Equating corresponding coordinates, we obtain the system

$$\begin{aligned}\alpha &= 4 \\ \alpha + \beta &= 2 \\ \beta &= -2.\end{aligned}$$

The first and third equations give $\alpha = 4$ and $\beta = -2$, and these values also satisfy the second equation. So $(4, 2, -2)$ belongs to $\langle S \rangle$ and it can be written as

$$(4, 2, -2) = 4(1, 1, 0) - 2(0, 1, 1).$$

Exercise C52

Let $\mathbf{v}_1 = (1, 0, 3)$, $\mathbf{v}_2 = (0, 2, 0)$ and $\mathbf{v}_3 = (0, 3, 1)$ be three vectors in \mathbb{R}^3 . Use Strategy C6 to determine whether the vector $(1, 5, 4)$ lies in the subset of \mathbb{R}^3 spanned by each of the following sets.

- (a) $\{\mathbf{v}_1, \mathbf{v}_2\}$ (b) $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$



Strategy C6 can also be used to show that a given set of vectors is a spanning set for the whole of a particular vector space, as we show in the following worked exercise.

Worked Exercise C30

Show that each of the following is a spanning set for \mathbb{R}^2 .

- (a) $\{(1, 2), (2, -3)\}$ (b) $\{(1, 0), (1, 1), (1, -2)\}$

Solution

 We need to show that *every* vector in \mathbb{R}^2 can be expressed as a linear combination of the given vectors, so we show that the general vector (x, y) can be. 

- (a) Each vector in \mathbb{R}^2 can be written as (x, y) . To show that (x, y) is in $\langle\{(1, 2), (2, -3)\}\rangle$, we write

$$(x, y) = \alpha(1, 2) + \beta(2, -3) = (\alpha + 2\beta, 2\alpha - 3\beta).$$

Equating corresponding coordinates, we obtain the system

$$\begin{aligned}\alpha + 2\beta &= x \\ 2\alpha - 3\beta &= y,\end{aligned}$$

whose solutions are $\alpha = \frac{1}{7}(3x + 2y)$, $\beta = \frac{1}{7}(2x - y)$. So any vector in \mathbb{R}^2 can be written in terms of $(1, 2)$ and $(2, -3)$ as

$$(x, y) = \frac{1}{7}(3x + 2y)(1, 2) + \frac{1}{7}(2x - y)(2, -3).$$

Thus $\{(1, 2), (2, -3)\}$ is a spanning set for \mathbb{R}^2 ; that is,



$$\langle\{(1, 2), (2, -3)\}\rangle = \mathbb{R}^2.$$

- (b) Each vector in \mathbb{R}^2 can be written as (x, y) . To show that (x, y) is in $\langle\{(1, 0), (1, 1), (1, -2)\}\rangle$, we write



$$\begin{aligned}(x, y) &= \alpha(1, 0) + \beta(1, 1) + \gamma(1, -2) \\ &= (\alpha + \beta + \gamma, \beta - 2\gamma).\end{aligned}$$

Equating corresponding coordinates, we obtain the system

$$\begin{aligned}\alpha + \beta + \gamma &= x \\ \beta - 2\gamma &= y.\end{aligned}$$

 We saw in Unit C1 that a consistent system of m equations in n unknowns, with $m < n$, has an infinite solution set. 

This is a system of two linear equations in three unknowns, so if there is a solution, there will be infinitely many solutions.

 We need just one solution, so try to simplify things by setting $\gamma = 0$. 

For example, taking $\gamma = 0$ gives $\beta = y$ and $\alpha = x - y$. So

$$(x, y) = (x - y)(1, 0) + y(1, 1) + 0(1, -2).$$

Thus $\langle\{(1, 0), (1, 1), (1, -2)\}\rangle = \mathbb{R}^2$.

The solution to Worked Exercise C30(b) shows that the set $\{(1, 0), (1, 1)\}$ is a spanning set for \mathbb{R}^2 so, in some sense, the vector $(1, -2)$ is redundant. We return to this idea of redundant vectors in a spanning set in the next section.

Exercise C53

Show that each of the following is a spanning set for \mathbb{R}^2 .

- (a) $\{(1, 1), (-1, 2)\}$ (b) $\{(2, -1), (3, 2)\}$

Exercise C54



Show that $\{(1, 0, 0), (1, 1, 0), (2, 0, 1)\}$ is a spanning set for \mathbb{R}^3 .

The following worked exercise shows that Strategy C6 can be used for vector spaces other than \mathbb{R}^2 and \mathbb{R}^3 .

Worked Exercise C31

Show that $\{1 + x^2, x^2, 2 - x\}$ is a spanning set for P_3 .

Solution

 As before, we need to show that *every* polynomial in P_3 can be expressed as a linear combination of the given polynomials, so we show that the general polynomial $a + bx + cx^2$ can be. 

Each polynomial in P_3 can be written as $a + bx + cx^2$. To show that $a + bx + cx^2$ is in $\langle \{1 + x^2, x^2, 2 - x\} \rangle$, we write

$$\begin{aligned} a + bx + cx^2 &= \alpha(1 + x^2) + \beta(x^2) + \gamma(2 - x) \\ &= \alpha + 2\gamma - \gamma x + (\alpha + \beta)x^2. \end{aligned}$$

Equating corresponding coefficients, we obtain the system

$$\begin{aligned} \alpha + 2\gamma &= a \\ -\gamma &= b \\ \alpha + \beta &= c. \end{aligned}$$

It follows from the second equation that $\gamma = -b$. Substituting this into the first equation gives $\alpha = a + 2b$ and hence, from the third equation, $\beta = c - a - 2b$. So

$$a + bx + cx^2 = (a + 2b)(1 + x^2) + (c - a - 2b)x^2 - b(2 - x).$$

Thus $\langle \{1 + x^2, x^2, 2 - x\} \rangle = P_3$.

Exercise C55

Show that $\{1 + x, 1 + x^2, 1 + x^3, x\}$ is a spanning set for P_4 .

We look now at sets S in vector spaces V for which $\langle S \rangle$ is not the whole of V .

Worked Exercise C32

For each of the following vector spaces V and sets of vectors S in V , determine $\langle S \rangle$. In parts (a) and (b), describe $\langle S \rangle$ geometrically.

(a) $V = \mathbb{R}^2$, $S = \{(1, 1)\}$.

(b) $V = \mathbb{R}^3$, $S = \{(1, 0, 1), (2, 0, 3)\}$.

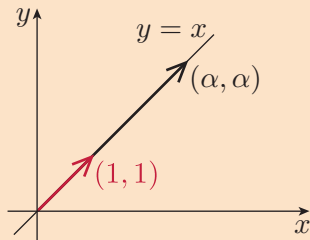
(c) $V = M_{2,3}$, $S = \left\{ \begin{pmatrix} 2 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 3 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -2 & 2 \\ 0 & 0 & 0 \end{pmatrix} \right\}$.

Solution

(a) We have

$$\langle S \rangle = \{\alpha(1, 1) : \alpha \in \mathbb{R}\} = \{(\alpha, \alpha) : \alpha \in \mathbb{R}\}.$$

☁ A picture can help. ☁



Geometrically, $\langle S \rangle$ is the line $y = x$.

(b) We have

$$\begin{aligned} \langle S \rangle &= \{\alpha(1, 0, 1) + \beta(2, 0, 3) : \alpha, \beta \in \mathbb{R}\} \\ &= \{(\alpha + 2\beta, 0, \alpha + 3\beta) : \alpha, \beta \in \mathbb{R}\}. \end{aligned}$$

☁ Every point in this set is of the form $(x, 0, z)$. ☁

Thus

$$\langle S \rangle \subseteq \{(x, 0, z) : x, z \in \mathbb{R}\}.$$

☁ To determine whether $\langle S \rangle$ is equal to this set we have to show that *every* vector $(x, 0, z)$ can be expressed as a linear combination of $(1, 0, 1)$ and $(2, 0, 3)$. ☁

To show that every vector $(x, 0, z)$, where $x, z \in \mathbb{R}$, belongs to $\langle S \rangle$, we write

$$(x, 0, z) = (\alpha + 2\beta, 0, \alpha + 3\beta).$$

Equating corresponding coordinates, we obtain the system

$$\alpha + 2\beta = x$$

$$\alpha + 3\beta = z.$$

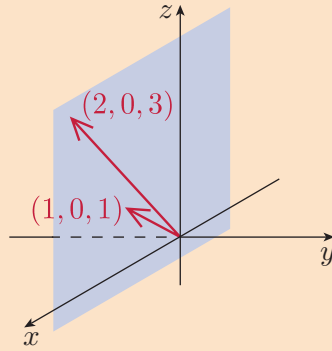
The solution is $\beta = z - x$ and $\alpha = 3x - 2z$, so

$$(x, 0, z) = (3x - 2z)(1, 0, 1) + (z - x)(2, 0, 3).$$

Hence $(x, 0, z) \in \langle S \rangle$, so any vector of the form $(x, 0, z)$ can be written in terms of $(1, 0, 1)$ and $(2, 0, 3)$. It follows that

$$\langle S \rangle = \{(x, 0, z) : x, z \in \mathbb{R}\}.$$

☁ A picture can help. ☁



Geometrically, $\langle S \rangle$ is the plane $y = 0$.

(c) We have

$$\begin{aligned} \langle S \rangle &= \left\{ \alpha \begin{pmatrix} 2 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \beta \begin{pmatrix} 1 & 0 & 3 \\ 0 & 0 & 0 \end{pmatrix} \right. \\ &\quad \left. + \gamma \begin{pmatrix} 0 & -2 & 2 \\ 0 & 0 & 0 \end{pmatrix} : \alpha, \beta, \gamma \in \mathbb{R} \right\} \\ &= \left\{ \begin{pmatrix} 2\alpha + \beta & -\alpha - 2\gamma & 3\beta + 2\gamma \\ 0 & 0 & 0 \end{pmatrix} : \alpha, \beta, \gamma \in \mathbb{R} \right\}. \end{aligned}$$

☁ Every matrix in this set is of the form $\begin{pmatrix} a & b & c \\ 0 & 0 & 0 \end{pmatrix}$. ☁

Thus

$$\langle S \rangle \subseteq \left\{ \begin{pmatrix} a & b & c \\ 0 & 0 & 0 \end{pmatrix} : a, b, c \in \mathbb{R} \right\}.$$

☁ To determine whether $\langle S \rangle$ is equal to this set we have to show that *every* matrix of this form can be expressed as a linear combination of the three given matrices. ☁

To show that every 2×3 matrix with zero entries in the second row belongs to $\langle S \rangle$, we write

$$\begin{pmatrix} a & b & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 2\alpha + \beta & -\alpha - 2\gamma & 3\beta + 2\gamma \\ 0 & 0 & 0 \end{pmatrix}.$$

Equating corresponding entries, we obtain the system

$$\begin{aligned} 2\alpha + \beta &= a \\ -\alpha - 2\gamma &= b \\ 3\beta + 2\gamma &= c. \end{aligned}$$

It has solution

$$\begin{aligned} \alpha &= \frac{1}{7}(3a - b - c), \\ \beta &= \frac{1}{7}(a + 2b + 2c), \\ \gamma &= -\frac{1}{14}(3a + 6b - c), \end{aligned}$$

so

$$\begin{pmatrix} a & b & c \\ 0 & 0 & 0 \end{pmatrix} \in \langle S \rangle.$$

Hence

$$\langle S \rangle = \left\{ \begin{pmatrix} a & b & c \\ 0 & 0 & 0 \end{pmatrix} : a, b, c \in \mathbb{R} \right\}.$$

Exercise C56

For each of the following vector spaces V and sets of vectors S in V , determine $\langle S \rangle$.

- (a) $V = \mathbb{R}^3$, $S = \{(1, 0, 0)\}$.
 (b) $V = M_{2,2}$, $S = \left\{ \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix} \right\}$.

3 Bases and dimension

In this section you will see that there is a minimum number of vectors needed to span a vector space.

3.1 Linear independence and dependence

In Section 2 we found several spanning sets for \mathbb{R}^2 and \mathbb{R}^3 . For example, in Worked Exercise C30(b), we showed that each of the sets

$$\{(1, 0), (1, 1)\} \quad \text{and} \quad \{(1, 0), (1, 1), (1, -2)\}$$

spans \mathbb{R}^2 . In order to be able to work efficiently with a vector space, we need to express each vector in it as a linear combination of a small number of vectors. In particular, it would be convenient if we could find a set containing the *smallest* number of vectors that spans the space – that is, we want to find a *minimal spanning set*.

The set $\{(1, 0), (1, 1), (1, -2)\}$ is clearly not a minimal spanning set for \mathbb{R}^2 , since the smaller set $\{(1, 0), (1, 1)\}$ also spans \mathbb{R}^2 . The vector $(1, -2)$ is redundant because it can be written as a linear combination of the vectors $(1, 0)$ and $(1, 1)$:

$$(1, -2) = 3(1, 0) - 2(1, 1).$$

Thus, if a vector (x, y) in \mathbb{R}^2 can be written as a linear combination of the vectors $(1, 0)$, $(1, 1)$ and $(1, -2)$, then it can be written as a linear combination of just the vectors $(1, 0)$ and $(1, 1)$:

$$\begin{aligned} (x, y) &= \alpha(1, 0) + \beta(1, 1) + \gamma(1, -2) \\ &= \alpha(1, 0) + \beta(1, 1) + \gamma[3(1, 0) - 2(1, 1)] \\ &= (\alpha + 3\gamma)(1, 0) + (\beta - 2\gamma)(1, 1). \end{aligned}$$

The following general result holds.

Theorem C20

Suppose that the vector \mathbf{v}_k can be written as a linear combination of the vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_{k-1}$. Then the span of the set $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ is the same as the span of the set $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_{k-1}\}$.

Proof Let $S = \langle \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_{k-1}\} \rangle$ and $T = \langle \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\} \rangle$.

Clearly, $S \subseteq T$.

Now

$$T = \{\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \dots + \alpha_k \mathbf{v}_k : \alpha_1, \alpha_2, \dots, \alpha_k \in \mathbb{R}\}.$$

As \mathbf{v}_k can be written as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_{k-1}$, it follows that

$$\mathbf{v}_k = \beta_1 \mathbf{v}_1 + \beta_2 \mathbf{v}_2 + \dots + \beta_{k-1} \mathbf{v}_{k-1}, \text{ for some } \beta_1, \beta_2, \dots, \beta_{k-1} \in \mathbb{R}.$$

So any vector of T can be expressed in the form

$$\begin{aligned} & \alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \cdots + \alpha_k \mathbf{v}_k \\ &= \alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \cdots + \alpha_{k-1} \mathbf{v}_{k-1} \\ & \quad + \alpha_k (\beta_1 \mathbf{v}_1 + \beta_2 \mathbf{v}_2 + \cdots + \beta_{k-1} \mathbf{v}_{k-1}) \\ &= (\alpha_1 + \alpha_k \beta_1) \mathbf{v}_1 + (\alpha_2 + \alpha_k \beta_2) \mathbf{v}_2 + \cdots + (\alpha_{k-1} + \alpha_k \beta_{k-1}) \mathbf{v}_{k-1}, \end{aligned}$$

which belongs to S . Thus $T \subseteq S$.

Combining these two results gives $S = T$, as required. ■

So, in order to tell whether a spanning set is minimal, we need to be able to test whether *every* vector in the set can be written as a linear combination of the remaining vectors in the set. To make this task easier, we introduce the ideas of *linear dependence* and *linear independence*.

Definitions

A finite set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ in a vector space V is **linearly dependent** if there exist real numbers $\alpha_1, \alpha_2, \dots, \alpha_k$, *not all zero*, such that

$$\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \cdots + \alpha_k \mathbf{v}_k = \mathbf{0}.$$

A finite set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ is **linearly independent** if it is not linearly dependent; that is, if

$$\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \cdots + \alpha_k \mathbf{v}_k = \mathbf{0}$$

only when $\alpha_1 = \alpha_2 = \cdots = \alpha_k = 0$.

Note that $\alpha_1 = \alpha_2 = \cdots = \alpha_k = 0$ is a solution to the equation whether the set of vectors is linearly dependent or linearly independent. So the distinction between the two cases is whether there is a *non-zero* solution.

We use the term *linearly dependent* because if a set of vectors is linearly dependent, then one of the vectors can be written as a linear combination of the others – that is, this vector *depends* on the others. If

$$\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \cdots + \alpha_k \mathbf{v}_k = \mathbf{0},$$

and α_k (for example) is non-zero, then we can rearrange the equation to give

$$\mathbf{v}_k = -\frac{\alpha_1}{\alpha_k} \mathbf{v}_1 - \cdots - \frac{\alpha_{k-1}}{\alpha_k} \mathbf{v}_{k-1},$$

so that \mathbf{v}_k is a linear combination of the remaining vectors. Hence $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ is a linearly dependent set.

For example, if $2\mathbf{v}_1 + 3\mathbf{v}_2 - 4\mathbf{v}_3 = \mathbf{0}$, then $\mathbf{v}_3 = \frac{1}{2}\mathbf{v}_1 + \frac{3}{4}\mathbf{v}_2$. In this case, $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a linearly dependent set. We can also write \mathbf{v}_1 in terms of \mathbf{v}_2 and \mathbf{v}_3 , and similarly \mathbf{v}_2 in terms of \mathbf{v}_1 and \mathbf{v}_3 .

Conversely, if one of a set of vectors can be written as a linear combination of the others, then the set is linearly dependent; that is, if \mathbf{v}_k is a linear combination of the vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_{k-1}$, then $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ is a linearly dependent set.

Statements 1 to 4 below follow from the definitions.

1. If $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ is a linearly independent set, then there is only one way in which the zero vector can be expressed as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$; that is, the trivial way

$$\mathbf{0} = 0\mathbf{v}_1 + 0\mathbf{v}_2 + \dots + 0\mathbf{v}_k.$$

2. If \mathbf{v}_1 is the zero vector, then for $\alpha \in \mathbb{R}$,

$$\alpha\mathbf{v}_1 + 0\mathbf{v}_2 + \dots + 0\mathbf{v}_k = \mathbf{0},$$

so any set of vectors containing the zero vector is linearly dependent. It follows that *a linearly independent set cannot contain the zero vector*.

3. Any set consisting of just one non-zero vector \mathbf{v} is linearly independent because if $\alpha\mathbf{v} = \mathbf{0}$, then either $\alpha = 0$ or $\mathbf{v} = \mathbf{0}$. Since \mathbf{v} is non-zero, we must have $\alpha = 0$, so the set $\{\mathbf{v}\}$ is linearly independent.
4. Any set of two non-zero vectors is linearly dependent if one of the vectors is a multiple of the other, and linearly independent otherwise. This applies to vectors in all vector spaces: it is not restricted to vectors in \mathbb{R}^2 and \mathbb{R}^3 .

As an example of statement 4, consider the set $\{(1, 1, 2), (2, 2, 4)\}$ in \mathbb{R}^3 . We have

$$(2, 2, 4) = 2(1, 1, 2),$$

so

$$-2(1, 1, 2) + (2, 2, 4) = (0, 0, 0),$$

which is the zero vector in \mathbb{R}^3 . In this case $\alpha_1 = -2$ and $\alpha_2 = 1$. So this set is linearly dependent.

Similarly, $\{3 - 2x + x^2, 6 - 4x + 2x^2\}$ is a linearly dependent set in P_3 because

$$6 - 4x + 2x^2 = 2(3 - 2x + x^2),$$

so

$$2(3 - 2x + x^2) - (6 - 4x + 2x^2) = 0 + 0x + 0x^2,$$

which is the zero vector in P_3 . In this case $\alpha_1 = 2$ and $\alpha_2 = -1$.

However, neither $\{(1, 1, 2), (1, 2, -3)\}$ nor $\{3 - 2x + x^2, -1 + x + 2x^2\}$ is a linearly dependent set, as in each case neither vector is a multiple of the other.

Statement 4 therefore gives us a particularly simple way of checking whether a set of two non-zero vectors is linearly dependent or linearly independent: namely, a set of two non-zero vectors is linearly independent if and only if neither vector is a multiple of the other. For vectors in \mathbb{R}^2 and \mathbb{R}^3 , this is equivalent to saying that two non-zero vectors are linearly independent if and only if they do not lie along the same straight line – that is, they are not *collinear*, as illustrated in Figure 7.

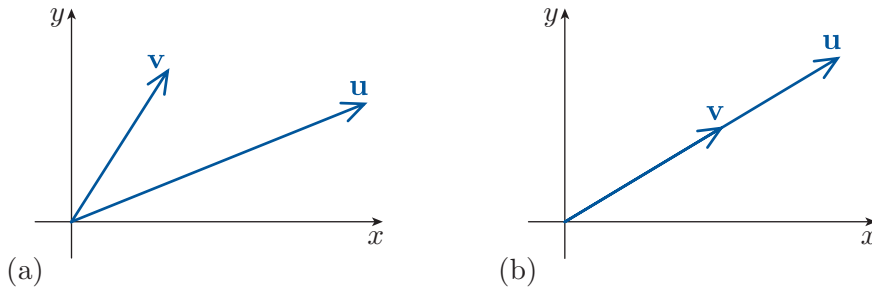


Figure 7 Two vectors in \mathbb{R}^2 that are (a) linearly independent (b) linearly dependent

In this geometric interpretation of \mathbb{R}^2 a vector (x, y) is the *position vector* (x, y) , not the point with coordinates (x, y) . Therefore ‘being collinear’ is a property of the vectors (position vectors), not the points with these coordinates. For example, the two *points* $(1, 0)$ and $(1, 1)$ are collinear since they lie on the line $x = 1$, whereas the *vectors* $(1, 0)$ and $(1, 1)$ are not collinear since they are not multiples of one another and they do not both lie on a line through the origin: they are linearly independent vectors. By their definition as position vectors, collinear vectors will always lie on a line through the origin.

Similarly, three non-zero vectors in \mathbb{R}^3 are linearly independent if and only if they do not lie in the same plane – that is, they are not *coplanar*, as illustrated in Figure 8. In this geometric interpretation of \mathbb{R}^3 ‘being coplanar’ is again a property of the vectors (position vectors) not the points, so coplanar vectors in \mathbb{R}^3 will always lie on a plane through the origin.

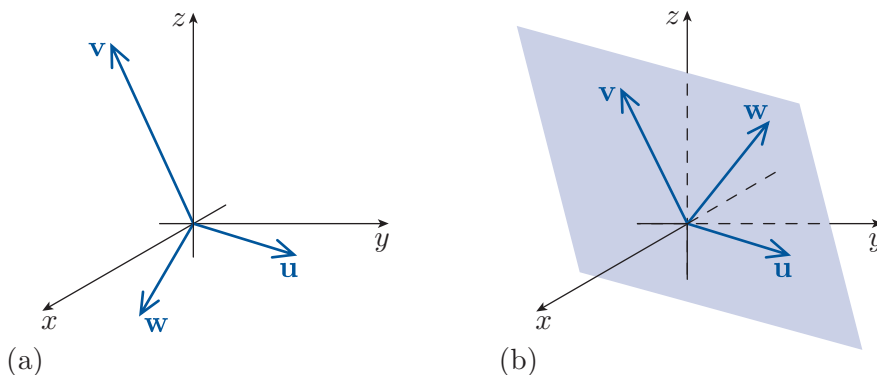


Figure 8 Three vectors in \mathbb{R}^3 that are (a) linearly independent (b) linearly dependent

More generally, we can use the following strategy to test whether a set of vectors is linearly independent.

Strategy C7

To test whether a given set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ is linearly independent:

1. write down the equation $\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \dots + \alpha_k \mathbf{v}_k = \mathbf{0}$
2. express this equation as a system of linear equations in the unknowns $\alpha_1, \alpha_2, \dots, \alpha_k$
3. solve the resulting system of equations.

If the only solution is $\alpha_1 = \alpha_2 = \dots = \alpha_k = 0$, then the set of vectors is linearly independent.

If there is a solution with at least one of $\alpha_1, \alpha_2, \dots, \alpha_k$ not equal to zero, then the set of vectors is linearly dependent.

Worked Exercise C33



Use Strategy C7 to determine whether each of the following sets of vectors in \mathbb{R}^3 is linearly independent.

- (a) $\{(2, 0, 0), (0, 0, 1), (-1, 2, 1)\}$ (b) $\{(1, 1, 1), (0, 2, 1), (1, 5, 3)\}$

Solution

We follow the steps of Strategy C7.

- (a) We write $\alpha(2, 0, 0) + \beta(0, 0, 1) + \gamma(-1, 2, 1) = (0, 0, 0)$.

 This simplifies to $(2\alpha - \gamma, 2\gamma, \beta + \gamma) = (0, 0, 0)$. Equating corresponding coordinates gives the equations we need. 

This gives the system of linear equations

$$\begin{aligned} 2\alpha - \gamma &= 0 \\ 2\gamma &= 0 \\ \beta + \gamma &= 0. \end{aligned}$$

The second equation gives $\gamma = 0$. Substituting this value into the other two equations gives $\alpha = 0$ and $\beta = 0$. The only solution is $\alpha = \beta = \gamma = 0$.

Therefore this set of vectors is linearly independent.

- (b) We write $\alpha(1, 1, 1) + \beta(0, 2, 1) + \gamma(1, 5, 3) = (0, 0, 0)$.

This gives the system of linear equations

$$\begin{aligned} \alpha + \gamma &= 0 \\ \alpha + 2\beta + 5\gamma &= 0 \\ \alpha + \beta + 3\gamma &= 0. \end{aligned}$$

☁ A solution is not so easy to see, so we use the method of Gauss–Jordan elimination from Unit C1. ☁

We perform row-reduction on the augmented matrix for this system of linear equations.

$$\begin{array}{l} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \mathbf{r}_3 \end{array} \quad \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 1 & 2 & 5 & 0 \\ 1 & 1 & 3 & 0 \end{array} \right) \begin{array}{l} 2 \\ 8 \\ 5 \end{array}$$

$$\begin{array}{l} \mathbf{r}_2 \rightarrow \mathbf{r}_2 - \mathbf{r}_1 \\ \mathbf{r}_3 \rightarrow \mathbf{r}_3 - \mathbf{r}_1 \end{array} \quad \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 2 & 4 & 0 \\ 0 & 1 & 2 & 0 \end{array} \right) \begin{array}{l} 2 \\ 6 \\ 3 \end{array}$$

$$\mathbf{r}_2 \rightarrow \frac{1}{2}\mathbf{r}_2 \quad \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 0 \end{array} \right) \begin{array}{l} 2 \\ 3 \\ 3 \end{array}$$

$$\mathbf{r}_3 \rightarrow \mathbf{r}_3 - \mathbf{r}_2 \quad \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right) \begin{array}{l} 2 \\ 3 \\ 0 \end{array}$$

The corresponding system of equations is

$$\begin{array}{rcl} \alpha & + & \gamma = 0 \\ \beta & + & 2\gamma = 0. \end{array}$$

The solution set of the system is

$$\alpha = -k, \quad \beta = -2k, \quad \gamma = k, \quad k \in \mathbb{R},$$

so there are infinitely many solutions. For example, $k = -1$ gives

$$(1, 1, 1) + 2(0, 2, 1) - (1, 5, 3) = (0, 0, 0).$$

So this set of vectors is linearly dependent.

☁ Any one of the vectors can be written as a linear combination of the other two, for example $(1, 1, 1) = (1, 5, 3) - 2(0, 2, 1)$. ☁

We claimed earlier that three non-zero linearly dependent vectors in \mathbb{R}^3 are coplanar and this was the case in Worked Exercise C33(b). You may like to check that all the vectors in the set lie in the plane through the origin with equation $x + y - 2z = 0$.

In the following exercise you are asked to determine whether given sets of vectors are linearly independent or not. Before embarking on the algebra, have a look at each set of vectors and try to decide whether you expect the set to be linearly dependent or linearly independent; it may be that Strategy C7 is not needed in some cases.

Exercise C57

Determine whether each of the following sets of vectors is a linearly independent subset of V .

- (a) $V = \mathbb{R}^2$, $\{(1, 0), (-1, -1)\}$.
- (b) $V = \mathbb{R}^2$, $\{(1, -1), (1, 1), (2, 1)\}$.
- (c) $V = \mathbb{R}^3$, $\{(1, 1, 0), (-1, 1, 1)\}$.
- (d) $V = \mathbb{R}^3$, $\{(1, 0, 0), (1, 1, 0), (1, 1, 1)\}$.
- (e) $V = \mathbb{R}^4$, $\{(1, 2, 1, 0), (0, -1, 1, 3)\}$.

We conclude this subsection by looking briefly at linearly dependent and linearly independent sets of vectors in vector spaces other than \mathbb{R}^2 , \mathbb{R}^3 and \mathbb{R}^4 . Again, before embarking on the algebra, it is sensible to have a look at each set of vectors: it may be that Strategy C7 is not needed in some cases.

Worked Exercise C34

Determine whether the set of polynomials $\{1, 4x, 4x + x^2\}$ is a linearly independent subset of P_3 .

Solution

 There is no obvious linear dependence. 

We apply Strategy C7.

We write $\alpha(1) + \beta(4x) + \gamma(4x + x^2) = 0$, which can be written as

$$\alpha + (4\beta + 4\gamma)x + \gamma x^2 = 0 + 0x + 0x^2.$$

Equating coefficients, we obtain the system

$$\begin{aligned}\alpha &= 0 \\ 4\beta + 4\gamma &= 0 \\ \gamma &= 0.\end{aligned}$$

So $\alpha = 0$, $\gamma = 0$ and, by substitution in the second equation, $\beta = 0$. The only solution is $\alpha = \beta = \gamma = 0$.

Therefore $\{1, 4x, 4x + x^2\}$ is a linearly independent subset of P_3 .



Worked Exercise C35

In each case, determine whether the set S of matrices is a linearly independent subset of $M_{2,2}$.


- (a) $S = \left\{ \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ -2 & 1 \end{pmatrix} \right\}$

- (b) $S = \left\{ \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} -2 & 2 \\ 0 & -4 \end{pmatrix} \right\}$
- (c) $S = \left\{ \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ -2 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 3 \\ 2 & 3 \end{pmatrix} \right\}$

Solution

- (a)  There are just two matrices and neither is a multiple of the other, so the strategy is unnecessary. 

The set S is linearly independent because neither matrix is a multiple of the other.

- (b)  The second matrix is a multiple of the first (-2 times), so the strategy is unnecessary. 

The set S is linearly dependent because

$$2 \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix} + \begin{pmatrix} -2 & 2 \\ 0 & -4 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

- (c)  There is no obvious linear dependence. 

We apply Strategy C7.

We write



$$\alpha \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} + \beta \begin{pmatrix} 0 & -1 \\ -2 & 1 \end{pmatrix} + \gamma \begin{pmatrix} 2 & 3 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

which can be written as

$$\begin{pmatrix} \alpha + 2\gamma & \alpha - \beta + 3\gamma \\ -2\beta + 2\gamma & 2\alpha + \beta + 3\gamma \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Equating corresponding entries, we obtain the system

$$\begin{aligned} \alpha &+ 2\gamma = 0 \\ \alpha - \beta + 3\gamma &= 0 \\ -2\beta + 2\gamma &= 0 \\ 2\alpha + \beta + 3\gamma &= 0. \end{aligned}$$

 The first and third equations both simply relate *two* unknowns, so it is sensible to start with these. 

From the third equation we have $2\beta = 2\gamma$, that is, $\beta = \gamma$, and from the first equation $\alpha = -2\gamma$. If we choose $\gamma = 1$, then $\beta = 1$ and $\alpha = -2$, and these also satisfy the second and fourth equations; thus

$$-2 \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} + 1 \begin{pmatrix} 0 & -1 \\ -2 & 1 \end{pmatrix} + 1 \begin{pmatrix} 2 & 3 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

So we can find α , β and γ not all zero such that the original equation is satisfied. So the set of matrices is linearly dependent. It is not a linearly independent subset of $M_{2,2}$.

Exercise C58

In each of the following cases, determine whether S is a linearly independent subset of the vector space V .

- (a) $V = P_4$, $S = \{1, x, x^2, x^3, 1 + x + x^2 + x^3\}$.
- (b) $V = M_{2,2}$, $S = \left\{ \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix} \right\}$.
- (c) $V = M_{2,2}$, $S = \left\{ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right\}$.
- (d) $V = \mathbb{C}$, $S = \{1 + i, 1 - i\}$.

3.2 Bases

We now use the idea of linear independence to help us find a minimal set of vectors that spans a vector space.

If we have a set of vectors that forms a spanning set for a vector space, then the set is a minimal spanning set if and only if *it is linearly independent*.

This condition is certainly necessary because, as we showed in the previous subsection, if the set of vectors is linearly dependent, then we can write at least one of the vectors as a linear combination of the other vectors. Such a vector is redundant, and we can drop it from the set, so the set is not a minimal set.

The condition is also sufficient; we prove this using proof by contradiction. Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ be a linearly independent spanning set for a vector space V , and suppose that the smaller set $S_1 = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_{k-1}\}$ also spans V . This means we can write any vector in V as a linear combination of the vectors in S_1 . In particular we can write

$$\mathbf{v}_k = \alpha_1 \mathbf{v}_1 + \cdots + \alpha_{k-1} \mathbf{v}_{k-1},$$

for some $\alpha_1, \dots, \alpha_{k-1}$ not all equal to 0. Therefore

$$\alpha_1 \mathbf{v}_1 + \cdots + \alpha_{k-1} \mathbf{v}_{k-1} - \mathbf{v}_k = \mathbf{0},$$

so S is not linearly independent. But this is a contradiction, so our initial assumption that S_1 spans V must be wrong. Thus S_1 cannot span V and S is a minimal spanning set.

If we have a linearly independent set of vectors that spans a vector space, then we give the set of vectors a special name.

Definition

A **basis** for a vector space V is a linearly independent set of vectors that is a spanning set for V .

The plural of basis is **bases**. A basis of a vector space V is *one* set of linearly independent vectors that spans V ; a basis is not unique, so V can have many different bases.

You saw in Exercise C53(a) that $\{(1, 1), (-1, 2)\}$ is a spanning set for \mathbb{R}^2 . Since it is also a linearly independent set, it is a *basis* for \mathbb{R}^2 . Although the set $\{(1, 0), (1, 1), (1, -2)\}$ is also a spanning set for \mathbb{R}^2 , it is not linearly independent, as we showed earlier in this section: so it is not a basis for \mathbb{R}^2 .

While each vector in \mathbb{R}^2 can be written as a linear combination of vectors in the spanning set $\{(1, 0), (1, 1), (1, -2)\}$, this expression is not unique.

For example,



$$\begin{aligned}(0, 1) &= 2(1, 0) - 1(1, 1) - 1(1, -2) \\ &= -4(1, 0) + 3(1, 1) + 1(1, -2).\end{aligned}$$

An important property of a basis for a vector space V is that each vector in V has a *unique* expression as a linear combination of basis vectors.

Theorem C21

Let S be a basis for a vector space V . Then each vector in V can be expressed as a linear combination of the vectors in S in only one way.

Proof Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ be a basis for a vector space V .

 We assume that a vector in V can be written as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ in two different ways, and show that this leads to a contradiction. 

Let \mathbf{u} be a vector in V , and assume that we can write \mathbf{u} as a linear combination of the vectors in S in two different ways as:

$$\mathbf{u} = \alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \cdots + \alpha_k \mathbf{v}_k$$


and

$$\mathbf{u} = \beta_1 \mathbf{v}_1 + \beta_2 \mathbf{v}_2 + \cdots + \beta_k \mathbf{v}_k.$$

Then

$$\mathbf{0} = \mathbf{u} - \mathbf{u} = (\alpha_1 - \beta_1)\mathbf{v}_1 + (\alpha_2 - \beta_2)\mathbf{v}_2 + \cdots + (\alpha_k - \beta_k)\mathbf{v}_k,$$

and $(\alpha_1 - \beta_1), (\alpha_2 - \beta_2), \dots, (\alpha_k - \beta_k)$ are not all zero.

Therefore the set S is linearly dependent. But S is a basis for V , and is therefore linearly independent. This contradiction shows that Theorem C21 is true. 

The definition of a basis gives us a strategy for testing whether a given set of vectors is a basis for a particular vector space.

Strategy C8

To determine whether a set of vectors S in a vector space V is a basis for V , check the following conditions.

- (1) S is linearly independent.
- (2) S spans V .

If both (1) and (2) hold, then S is a basis for V .

If either (1) or (2) does not hold, then S is not a basis for V .

Worked Exercise C36

Show that $S = \{(2, 0, 2), (1, 1, 1), (0, 1, -1)\}$ is a basis for \mathbb{R}^3 .

Solution

We check both conditions in Strategy C8.

 We start by checking condition (1): S is linearly independent. 

Using Strategy C7, we write



$$\alpha(2, 0, 2) + \beta(1, 1, 1) + \gamma(0, 1, -1) = (0, 0, 0),$$

which simplifies to

$$(2\alpha + \beta, \beta + \gamma, 2\alpha + \beta - \gamma) = (0, 0, 0).$$

Equating corresponding coordinates, we obtain the system

$$\begin{aligned} 2\alpha + \beta &= 0 \\ \beta + \gamma &= 0 \\ 2\alpha + \beta - \gamma &= 0. \end{aligned}$$



 We could use Gauss–Jordan elimination, but we can solve this system directly. 

Subtracting the third equation from the first gives $\gamma = 0$, and substituting this into the second equation gives $\beta = 0$. Finally, substituting $\beta = 0$ into the first equation gives $\alpha = 0$. The only solution is $\alpha = \beta = \gamma = 0$.

Therefore the set S is linearly independent.

 We now check condition (2): S spans \mathbb{R}^3 . 

We apply Strategy C6.

 We need to show that *every* vector in \mathbb{R}^3 can be expressed as a linear combination of the vectors in S , so we show that the general vector (x, y, z) can be. 

Each vector in \mathbb{R}^3 can be written as (x, y, z) , with $x, y, z \in \mathbb{R}$. To show that (x, y, z) is in $\langle S \rangle$, we write

$$(x, y, z) = \alpha(2, 0, 2) + \beta(1, 1, 1) + \gamma(0, 1, -1).$$

Equating corresponding coordinates, we obtain the system

$$\begin{aligned} 2\alpha + \beta &= x \\ \beta + \gamma &= y \\ 2\alpha + \beta - \gamma &= z. \end{aligned}$$

Subtracting the third equation from the first gives $\gamma = x - z$, and substituting this into the second equation gives $\beta = y - x + z$. Finally, substituting for β in the first equation gives $\alpha = \frac{1}{2}(2x - y - z)$. We have a solution, so any vector in \mathbb{R}^3 can be written in terms of vectors in S as

$$\begin{aligned} (x, y, z) &= \frac{1}{2}(2x - y - z)(2, 0, 2) + (y - x + z)(1, 1, 1) \\ &\quad + (x - z)(0, 1, -1). \end{aligned}$$

Therefore S spans \mathbb{R}^3 .

Since conditions (1) and (2) hold, the set S is a basis for \mathbb{R}^3 .

Worked Exercise C37

Determine whether each of the following sets is a basis for \mathbb{R}^3 .



- (a) $\{(0, 1, 2), (1, 2, -1)\}$ (b) $\{(1, 1, 1), (0, 2, 1), (-1, 1, 0)\}$

Solution

- (a) We check both conditions in Strategy C8.

The set $\{(0, 1, 2), (1, 2, -1)\}$ is linearly independent, as neither vector is a multiple of the other.

We apply Strategy C6.

 We need to show that *every* vector in \mathbb{R}^3 can be expressed as a linear combination of the given vectors, so we show that the general vector can be. 

Each vector in \mathbb{R}^3 can be written as (x, y, z) , with $x, y, z \in \mathbb{R}$. To show that (x, y, z) is in $\langle \{(0, 1, 2), (1, 2, -1)\} \rangle$, we write

$$(x, y, z) = \alpha(0, 1, 2) + \beta(1, 2, -1).$$

Equating corresponding coordinates, we obtain the system

$$\begin{aligned} \beta &= x \\ \alpha + 2\beta &= y \\ 2\alpha - \beta &= z. \end{aligned}$$

Substituting $\beta = x$ from the first equation into the other two equations gives

$$\alpha = y - 2x$$

$$\alpha = \frac{1}{2}(x + z).$$

☁ The vector (x, y, z) is a general vector, so we need a solution for every possible combination of x , y and z . ☁

These two equations are true simultaneously if and only if $y - 2x = \frac{1}{2}(x + z)$; that is, if and only if $5x - 2y + z = 0$.

☁ This is not true for every x , y and z . In fact, it shows that $\langle \{(0, 1, 2), (1, 2, -1)\} \rangle$ is the plane $5x - 2y + z = 0$ in \mathbb{R}^3 ; thus any point not on this plane cannot be written as a linear combination of the vectors $(0, 1, 2)$ and $(1, 2, -1)$. ☁

This contradicts the assumption that x , y and z can take any real values, so $\{(0, 1, 2), (1, 2, -1)\}$ is not a spanning set for \mathbb{R}^3 . Thus it is not a basis for \mathbb{R}^3 .

(b) We check both conditions in Strategy C8.

☁ Before diving into Strategy C7, we quickly look at the given vectors to see if there is any obvious linear dependence. ☁

Here we have

$$(-1, 1, 0) = -(1, 1, 1) + (0, 2, 1),$$

so these vectors are not linearly independent.

Therefore the set $\{(1, 1, 1), (0, 2, 1), (-1, 1, 0)\}$ is not a basis for \mathbb{R}^3 .

Exercise C59

Determine whether each of the following sets is a basis for \mathbb{R}^3 .

- (a) $\{(0, 1, 2), (0, 2, 3), (0, 6, 1)\}$
- (b) $\{(1, 2, 1), (1, 0, -1), (0, 3, 1)\}$
- (c) $\{(1, 0, 0), (0, 1, 0), (0, 0, 1), (1, 1, 1)\}$

Exercise C60

Determine whether $\{(1, 2, -1, -1), (-1, 5, 1, 3)\}$ is a basis for \mathbb{R}^4 .

We now consider bases for vector spaces other than \mathbb{R}^2 , \mathbb{R}^3 and \mathbb{R}^4 .



Worked Exercise C38

Determine whether each of the following sets is a basis for P_3 .

- (a) $\{1, x, x^2\}$ (b) $\{1, x\}$ (c) $\{1, 2 + x^2, x^2\}$

Solution

- (a) We check both conditions in Strategy C8.

 We check whether $\{1, x, x^2\}$ is linearly independent. 



Using Strategy C7, we write

$$\alpha 1 + \beta x + \gamma x^2 = 0 + 0x + 0x^2.$$

Comparing coefficients, we have $\alpha = \beta = \gamma = 0$ as the only solution, so the set is linearly independent.

 We check whether $\{1, x, x^2\}$ spans P_3 . 

We apply Strategy C6.

 We need to show that *every* vector (polynomial) in P_3 can be written as a linear combination of 1, x and x^2 , so we show that the general vector $a + bx + cx^2$ can be. 



Each vector in P_3 can be written as $a + bx + cx^2$, with $a, b, c \in \mathbb{R}$. To show that $a + bx + cx^2$ is in $\langle \{1, x, x^2\} \rangle$, we write

$$a + bx + cx^2 = \alpha(1) + \beta(x) + \gamma(x^2).$$

Equating coefficients, we see that $a = \alpha$, $b = \beta$ and $c = \gamma$.



Therefore the set of vectors spans P_3 .

Thus $\{1, x, x^2\}$ is a basis for P_3 .

- (b)  Notice that x^2 cannot be expressed as a linear combination of 1 and x . 

None of the vectors contains an x^2 term, so the set $\{1, x\}$ does not span P_3 .

Therefore this set of vectors is not a basis for P_3 .



 You may have noticed that neither vector is a multiple of the other, so the set $\{1, x\}$ is linearly independent. The span of this set consists of polynomials of the form $a + bx$, which is a proper subset of P_3 . 

- (c) Here we have

$$2 + x^2 = 2(1) + 1(x^2),$$

so the set $\{1, 2 + x^2, x^2\}$ is not linearly independent.

Therefore $\{1, 2 + x^2, x^2\}$ is not a basis for P_3 .

 The span of this set consists of all polynomials of the form $a + bx^2$, which again is a proper subset of P_3 . 

Exercise C61

Determine whether

$$S = \left\{ \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -3 & 1 \\ 0 & 0 \end{pmatrix} \right\}$$

is a basis for $M_{2,2}$.

3.3 Standard bases

You may have noticed that some sets of basis vectors seem to make the calculations in vector spaces particularly simple. For \mathbb{R}^2 this set is $\{(1, 0), (0, 1)\}$, for \mathbb{R}^3 it is $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$, and so on.

The representation of a vector in terms of these bases is straightforward. For example, in \mathbb{R}^2

$$(x, y) = x(1, 0) + y(0, 1),$$

and in \mathbb{R}^3

$$(x, y, z) = x(1, 0, 0) + y(0, 1, 0) + z(0, 0, 1).$$

Because these bases are so simple, they are used frequently; they are called *standard bases*.

Definition

The **standard basis** for \mathbb{R}^n is the set of n vectors

$$\{(1, 0, \dots, 0), (0, 1, 0, \dots, 0), \dots, (0, \dots, 0, 1)\}.$$

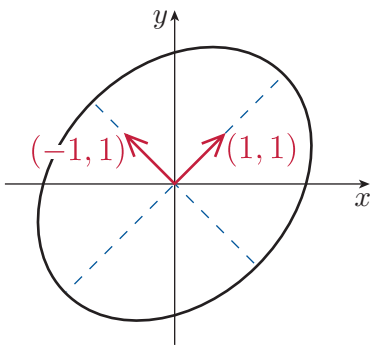


Figure 9 An ellipse with non-standard basis shown

The standard basis for \mathbb{R}^n seems so natural that you may wonder why we do not use it all the time. In some physical situations, however, we may need to choose a different basis. For example, if we are looking at an ellipse centred at the origin, we may want to choose basis vectors along the major and minor axes of the ellipse. For the ellipse shown in Figure 9, it may be more convenient to choose the basis vectors $(1, 1)$ and $(-1, 1)$ rather than the standard ones, $(1, 0)$ and $(0, 1)$. Similarly, if we are considering a parallelogram, we may want to choose basis vectors along the sides of the parallelogram. In many vector spaces other than \mathbb{R}^n there are particularly simple bases, which we call the standard bases for these spaces. Here are some examples.

$$P_n : \{1, x, x^2, \dots, x^{n-1}\}$$

$$M_{2,2} : \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

$$\mathbb{C} : \{1, i\}$$

If we write a vector in \mathbb{R}^2 as (x, y) , then x and y are the components, or *coordinates*, of the vector with respect to the standard basis vectors – that is,

$$(x, y) = x(1, 0) + y(0, 1).$$

However, we need some way of indicating what the *coordinates* of a vector are with respect to non-standard basis vectors. We use the following notation.

Definitions

Let $E = \{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ be a basis for a vector space V , and suppose that

$$\mathbf{v} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + \dots + v_n\mathbf{e}_n,$$

where $v_1, v_2, \dots, v_n \in \mathbb{R}$.

Then the ***E*-coordinate representation** of \mathbf{v} is

$$\mathbf{v}_E = (v_1, v_2, \dots, v_n)_E.$$

We call v_1, v_2, \dots, v_n the **coordinates of \mathbf{v} with respect to the basis E** , or, more briefly, the ***E*-coordinates** of \mathbf{v} .

Remarks

1. We usually omit the subscript if E is the standard basis.
2. We write the basis vectors as $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ rather than $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ to avoid confusion between the basis vectors and the coordinates v_1, v_2, \dots, v_n of a vector \mathbf{v} .
3. We can denote the E -coordinates of a vector \mathbf{v}_j by $v_{1j}, v_{2j}, \dots, v_{nj}$. So we write $\mathbf{v}_j = v_{1j}\mathbf{e}_1 + v_{2j}\mathbf{e}_2 + \dots + v_{nj}\mathbf{e}_n$.
4. Since E is a basis for V , the E -coordinate representation of a vector in V is unique. However, the order of the coordinates in such a representation depends on the order of the basis vectors.
5. A non-zero vector has a different coordinate representation for each different basis. For the zero vector, the coordinates are always zero.
You can think of the different representations of a vector as analogous to an amount of money being expressed in different currencies; in every currency, ‘no money’ is the same as ‘zero money’.
6. If E is a standard basis, then we refer to the *standard coordinate representation*, *standard coordinates*, and so on.

The following worked exercise shows this notation in practice.

Worked Exercise C39

Given the basis $E = \{(-1, 2), (2, 2)\}$ for \mathbb{R}^2 , determine the standard coordinate representation of $(3, 2)_E$.

Solution

The coordinates $(3, 2)_E$ are with respect to the basis E , meaning three times the first basis vector in E and twice the second.

For the basis $E = \{(-1, 2), (2, 2)\}$, we have

$$\begin{aligned}(3, 2)_E &= 3(-1, 2) + 2(2, 2) \\ &= (-3, 6) + (4, 4) \\ &= (1, 10).\end{aligned}$$

There is no subscript on the coordinates $(1, 10)$ because they are with respect to the standard basis for \mathbb{R}^2 .

Exercise C62

- (a) Given the basis $E = \{(1, 2), (-3, 1)\}$ for \mathbb{R}^2 , determine the standard coordinate representation of $(2, 1)_E$.
- (b) Given the basis $E = \{(1, 0, 2), (-1, 1, 3), (2, -2, 0)\}$ for \mathbb{R}^3 , determine the standard coordinate representation of $(1, 1, -1)_E$.

We can also turn around the method in Worked Exercise C39 to express a given vector in terms of a non-standard basis.

Worked Exercise C40

For each of the following bases E for \mathbb{R}^2 , find the E -coordinate representation of the vector $(1, 4)$.

- (a) $E = \{(1, 4), (4, -1)\}$ (b) $E = \{(-1, 2), (2, 2)\}$

Solution

- (a) We write $(1, 4) = \alpha(1, 4) + \beta(4, -1)$, which has the solution $\alpha = 1, \beta = 0$, so

$$(1, 4) = 1(1, 4) + 0(4, -1) = (1, 0)_E.$$

(b) We write $(1, 4) = \alpha(-1, 2) + \beta(2, 2) = (-\alpha + 2\beta, 2\alpha + 2\beta)$.

Equating corresponding coordinates, we obtain the system

$$\begin{aligned} -\alpha + 2\beta &= 1 \\ 2\alpha + 2\beta &= 4. \end{aligned}$$

Solving these equations gives $\alpha = 1$ and $\beta = 1$, so

$$(1, 4) = 1(-1, 2) + 1(2, 2) = (1, 1)_E.$$

Geometrically, by changing the basis we are changing the axes we are using. For example, in Worked Exercise C40(b) we are expressing the vector $(1, 4)$ (with respect to the standard basis) as a vector in terms of the new basis vectors $E = \{(-1, 2), (2, 2)\}$. The E -coordinates of this vector with respect to the basis E are $(1, 1)_E$ representing one step along the $(-1, 2)$ -axis then one step along the $(2, 2)$ -axis. Figure 10 illustrates how this vector is represented with respect to these new axes.

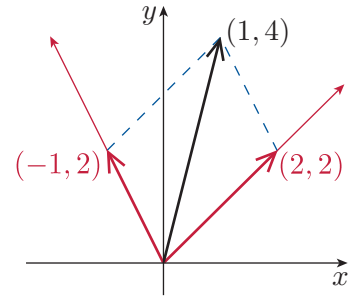


Figure 10 Changing the axes

Worked Exercise C41

Find the E -coordinate representation of the vector $(-2, 0, 1)$ with respect to the basis $E = \{(1, 0, 0), (1, 0, 1), (2, 1, -1)\}$ for \mathbb{R}^3 .

Solution

We write

$$\begin{aligned} (-2, 0, 1) &= \alpha(1, 0, 0) + \beta(1, 0, 1) + \gamma(2, 1, -1) \\ &= (\alpha + \beta + 2\gamma, \gamma, \beta - \gamma). \end{aligned}$$

Equating corresponding coordinates, we obtain the system

$$\begin{aligned} \alpha + \beta + 2\gamma &= -2 \\ \gamma &= 0 \\ \beta - \gamma &= 1. \end{aligned}$$

The second equation gives $\gamma = 0$. Substituting this value into the third equation gives $\beta = 1$, and substituting these values into the first equation gives $\alpha = -3$. So

$$\begin{aligned} (-2, 0, 1) &= -3(1, 0, 0) + 1(1, 0, 1) + 0(2, 1, -1) \\ &= (-3, 1, 0)_E. \end{aligned}$$

Exercise C63

- (a) Find the E -coordinate representation of the vector $(5, -4)$ with respect to the basis $E = \{(1, 2), (-3, 1)\}$ for \mathbb{R}^2 .
- (b) Find the E -coordinate representation of the vector $(-3, 5, 7)$ with respect to the basis $E = \{(1, 0, 2), (-1, 1, 3), (2, -2, 0)\}$ for \mathbb{R}^3 .

3.4 Dimension

You may have noticed in the previous subsection that all the bases you met for \mathbb{R}^2 contained two vectors, all the bases for \mathbb{R}^3 contained three vectors, and so on. This should correspond to your intuitive idea of dimension – namely that \mathbb{R} is one-dimensional, \mathbb{R}^2 is two-dimensional, and so on.

For example, among the bases you met were the following.

$$\mathbb{R}^2 : \{(1, 0), (0, 1)\}, \quad \{(1, 0), (1, 1)\}, \quad \{(1, 2), (-1, 1)\}.$$

$$\mathbb{R}^3 : \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}, \quad \{(1, 2, 1), (1, 0, -1), (0, 3, 1)\}.$$

$$\mathbb{R}^4 : \{(1, 0, 2, 0), (0, 1, 0, 3), (0, 0, 1, 2), (2, 0, -1, 0)\}, \\ \{(1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1)\}.$$

It is not a coincidence that every basis for \mathbb{R}^2 contains exactly two vectors, and every basis for \mathbb{R}^3 contains exactly three vectors. The main theorem in this section, the *Basis Theorem*, states that if V is any vector space, then *every basis for V contains the same number of vectors*. Before we prove this, we must define what we mean by a finite-dimensional vector space.

Definitions



Let V be a vector space. Then V is **finite-dimensional** if it contains a finite set of vectors S that forms a basis for V . If no such set exists, then V is **infinite-dimensional**.

Examples of infinite-dimensional vector spaces are \mathbb{R}^∞ and the set of polynomials of any degree. On the other hand, the set containing just the zero vector is a zero-dimensional vector space, which has the empty set as its basis.

In order to prove that every basis for a finite-dimensional vector space V contains the same number of vectors, we first prove the following useful result.

Theorem C22

Let $E = \{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ be a basis for a vector space V , and let $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m\}$ be a set of m vectors in V , where $m > n$. Then S is a linearly dependent set.

Proof  We assume that the conditions of Theorem C22 hold and show that this implies that S is linearly dependent. 

Let $E = \{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ be a basis for V , and let $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m\}$ be a set of m vectors in V . Then each of the vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m$ can be written as a linear combination of the vectors in E ; that is,

$$\mathbf{v}_1 = v_{11}\mathbf{e}_1 + v_{21}\mathbf{e}_2 + \dots + v_{n1}\mathbf{e}_n,$$

$$\mathbf{v}_2 = v_{12}\mathbf{e}_1 + v_{22}\mathbf{e}_2 + \dots + v_{n2}\mathbf{e}_n,$$

$$\vdots$$

$$\mathbf{v}_m = v_{1m}\mathbf{e}_1 + v_{2m}\mathbf{e}_2 + \dots + v_{nm}\mathbf{e}_n,$$

for some numbers $v_{11}, \dots, v_{nm} \in \mathbb{R}$.

To show that S is linearly dependent, we must find real numbers $\alpha_1, \alpha_2, \dots, \alpha_m$, not all zero, such that

$$\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \dots + \alpha_m\mathbf{v}_m = \mathbf{0}. \quad (1)$$

Using the first system of equations, we can rewrite equation (1) as

$$\begin{aligned} & (\alpha_1v_{11} + \alpha_2v_{12} + \dots + \alpha_mv_{1m})\mathbf{e}_1 \\ & + (\alpha_1v_{21} + \alpha_2v_{22} + \dots + \alpha_mv_{2m})\mathbf{e}_2 \\ & + \dots + (\alpha_1v_{n1} + \alpha_2v_{n2} + \dots + \alpha_mv_{nm})\mathbf{e}_n = \mathbf{0}. \end{aligned} \quad (2)$$

Since E is a basis, the set of vectors $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ is linearly independent. It follows that we can find real numbers $\alpha_1, \alpha_2, \dots, \alpha_m$, not all zero, that satisfy equation (2) if and only if the following system of equations has a non-zero solution for $\alpha_1, \alpha_2, \dots, \alpha_m$:



$$v_{11}\alpha_1 + v_{12}\alpha_2 + \dots + v_{1m}\alpha_m = 0$$


$$v_{21}\alpha_1 + v_{22}\alpha_2 + \dots + v_{2m}\alpha_m = 0$$

$$\vdots$$

$$v_{n1}\alpha_1 + v_{n2}\alpha_2 + \dots + v_{nm}\alpha_m = 0.$$

This is a system of n linear equations in m unknowns with $m > n$, so there are more unknowns than equations.

 In Unit C1 you saw that a consistent system with more unknowns than equations has an infinite solution set. The system above is consistent because it is homogeneous, and therefore it has an infinite solution set. 

Such a system of linear equations has a non-trivial solution – that is, a solution for which some variables are non-zero. Therefore the set S containing $m > n$ vectors is linearly dependent. This proves the theorem. 

For example, \mathbb{R}^3 has three vectors in its standard basis, so, by Theorem C22, the set

$$\{(1, 1, 0), (0, -2, 1), (0, 0, 1), (1, 1, 2)\}$$

is linearly dependent because it contains more than three vectors. In fact,

$$(1, 1, 0) + 0(0, -2, 1) + 2(0, 0, 1) - (1, 1, 2) = (0, 0, 0).$$

Theorem C22 has the following immediate, and useful, consequence.



Corollary C23

Let V be a vector space with a basis containing n vectors. If a linearly independent subset of V contains m vectors, then $m \leq n$.

This corollary provides the crucial steps in the proof of the Basis Theorem.

Theorem C24 Basis Theorem

Let V be a finite-dimensional vector space. Then every basis for V contains the same number of vectors.

Proof  We assume there are two bases with n and m vectors, respectively, and show that since a basis is a linearly independent set, this implies that $n = m$. 

Let $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ and $\{\mathbf{f}_1, \mathbf{f}_2, \dots, \mathbf{f}_m\}$ be two bases for a finite-dimensional vector space V .

Since $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ is a basis for V and $\{\mathbf{f}_1, \mathbf{f}_2, \dots, \mathbf{f}_m\}$ is a linearly independent set, we have $m \leq n$, by Corollary C23.

Similarly, since $\{\mathbf{f}_1, \mathbf{f}_2, \dots, \mathbf{f}_m\}$ is a basis for V and $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ is linearly independent, we have $n \leq m$, by Corollary C23.

Therefore $m = n$, so every basis contains the same number of vectors. 

The Basis Theorem allows us to give a definition of the dimension of a finite-dimensional vector space, which agrees with our intuitive idea of dimension.

Definition

The **dimension** of a finite-dimensional vector space V , denoted by $\dim V$, is the number of vectors in any basis for the space.

So \mathbb{R}^2 has dimension 2 and \mathbb{R}^3 has dimension 3, as we would expect. More generally, \mathbb{R}^n has dimension n , since the standard basis for \mathbb{R}^n has n vectors. It follows from Theorem C24 that every basis for \mathbb{R}^n contains exactly n vectors. The strategy for checking whether a set of vectors is a basis (Strategy C8) can now be greatly simplified when the vector space is \mathbb{R}^n . The result that we need is stated in the next theorem.

Theorem C25

Let V be an n -dimensional vector space. Then any set of n linearly independent vectors in V is a basis for V .

Proof  We give a proof by contradiction. 


Suppose that the set $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ of n linearly independent vectors does not span V . Then there exists a vector \mathbf{v} in V that cannot be written as a linear combination of the vectors in S .

So, if

$$\alpha_1 \mathbf{v}_1 + \cdots + \alpha_n \mathbf{v}_n + \alpha_{n+1} \mathbf{v} = \mathbf{0},$$

then $\alpha_{n+1} = 0$, since \mathbf{v} cannot be written as a linear combination of the vectors in S and $\alpha_1 = \cdots = \alpha_n = 0$, since S is linearly independent. Hence $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n, \mathbf{v}\}$ is a linearly independent set of vectors.

But by Theorem C22, any set of more than n vectors is linearly dependent. This is a contradiction so the original statement must be false, and S does span V .

Therefore every set of n linearly independent vectors in V is a basis for V . 

This means that to check whether a set S is a basis for \mathbb{R}^n , we no longer have to check that S spans \mathbb{R}^n : we *know* that it does if it is linearly independent and contains n vectors. We can simplify Strategy C8.

In fact, we can use this simplified strategy to determine whether a set of vectors is a basis for any vector space V if we know the dimension of V .

Strategy C9

To determine whether a set of vectors S in a vector space V of dimension n is a basis, check the following conditions.

- (1) S contains n vectors.
- (2) S is linearly independent.

If both (1) and (2) hold, then S is a basis for V .

If either (1) or (2) does not hold, then S is not a basis for V .

Exercise C64

Use Strategy C9 to determine which of the following sets is a basis for \mathbb{R}^3 .

- (a) $\{(1, 2, 1), (1, 0, -1)\}$ (b) $\{(1, 0, 1), (1, 0, -1), (0, 1, 1)\}$
 (c) $\{(1, -1, 0), (2, 1, 4), (3, 0, 4)\}$
 (d) $\{(1, 0, 0), (0, 1, 0), (0, 0, 1), (1, 1, 1)\}$

Strategy C9 is easier to use than Strategy C8 because you can eliminate sets that do not contain the right number of vectors. Furthermore, you do not need to check spanning, which is usually harder than checking for linear independence.

To be able to apply Strategy C9 to vector spaces other than \mathbb{R}^n we need to know the dimension of other vector spaces.

In Subsection 3.3 we listed the standard bases for some vector spaces as follows.

$$\mathbb{R}^n : \quad \{(1, 0, \dots, 0), (0, 1, 0, \dots, 0), \dots, (0, \dots, 0, 1)\}.$$

$$P_n : \quad \{1, x, x^2, \dots, x^{n-1}\}.$$

$$M_{2,2} : \quad \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}.$$

$$\mathbb{C} : \quad \{1, i\}.$$

We can see that the dimension of P_n is n , so the dimension of P_2 is 2, the dimension of P_3 is 3, and so on.

Similarly, the dimension of $M_{2,2}$ is 4, and, in general, the dimension of $M_{m,n}$ is mn . For example, $M_{2,3}$ has dimension 6: a basis is

$$\left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}.$$

Finally, the dimension of \mathbb{C} is 2.

Exercise C65

Use Strategy C9 to determine whether each of the following sets is a basis for the given vector space.

- (a) The set S for $M_{2,2}$, where

$$S = \left\{ \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \right\}$$

- (b) The set $S = \{2 + x, 1 - x\}$ for P_2 .

We end this section by showing that a linearly independent subset of a vector space can always be extended to give a basis for the vector space. This result will be useful in Unit C3 *Linear transformations*.

Theorem C26

Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m\}$ be a linearly independent subset of an n -dimensional vector space V , where $m < n$. Then there exist vectors $\mathbf{v}_{m+1}, \dots, \mathbf{v}_n$ in V such that $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is a basis for V .

Proof Since $m < n$, S is not a basis for V , by the Basis Theorem (Theorem C24) and Theorem C25. Thus there is a vector \mathbf{v}_{m+1} in V that cannot be expressed as a linear combination of the vectors in S . As in the proof of Theorem C25, it follows that $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_{m+1}\}$ is linearly independent.

We keep adding vectors in this way until we obtain a linearly independent set with n vectors. This is a basis, by Theorem C25. ■

4 Subspaces

In this section you will meet subsets of vector spaces that are themselves vector spaces.

4.1 Definition

You have seen examples where a set of vectors does not span the whole of a vector space, but spans only a proper subset of that vector space, for example in Worked Exercise C32 and Exercise C56. In particular, you saw the following.

- In \mathbb{R}^2 , the set of vectors $\{(1, 1)\}$ is a spanning set for the line through the origin with equation $y = x$; this is a one-dimensional subset of \mathbb{R}^2 .
- In \mathbb{R}^3 , the set of vectors $\{(1, 0, 0)\}$ is a spanning set for the x -axis; this is a one-dimensional subset of \mathbb{R}^3 .
- In \mathbb{R}^3 , the set of vectors $\{(1, 0, 1), (2, 0, 3)\}$ is a spanning set for the plane $y = 0$; this is a two-dimensional subset of \mathbb{R}^3 .

In fact, any proper subset of \mathbb{R}^3 that is the span of a set of vectors must take one of the following forms: $\{\mathbf{0}\}$, a line through the origin (a one-dimensional subset), or a plane through the origin (a two-dimensional subset).

When you met these examples, you may have asked yourself whether these subsets are themselves vector spaces. In fact, they are; we call such subsets *subspaces*.

Definition

A subset S of a vector space V is a **subspace** of V if S is itself a vector space under vector addition and scalar multiplication as defined in V .

In order to prove that a subset S is a vector space, we must show that it satisfies all the axioms in Subsection 1.2. In practice, however, we do not need to check them all, as many of them carry over from V ; that is, if they are true for V , then they are also true for S . For example, the commutativity axiom (A5) states that $\mathbf{v}_1 + \mathbf{v}_2 = \mathbf{v}_2 + \mathbf{v}_1$, for all $\mathbf{v}_1, \mathbf{v}_2 \in V$; since all the vectors in S are also in V , this axiom holds for S .

Provided that S is non-empty, the only axioms that need to be checked are the closure axioms (A1 and S1), because all the other axioms follow from V . If the zero vector is in S , then S is non-empty. Therefore we can replace the condition that S is non-empty by the condition that the zero vector is in S . This gives the following theorem; you are asked to prove this as an exercise in the additional exercises booklet for this unit.

Theorem C27

A subset S of a vector space V is a subspace of V if it satisfies the following conditions.

- (a) $\mathbf{0} \in S$.
- (b) S is closed under vector addition.
- (c) S is closed under scalar multiplication.

This theorem allows us to give a strategy for testing whether a given subset of a vector space is a subspace.

Strategy C10

To test whether a given subset S of a vector space V is a subspace of V , check the following conditions.

- (1) $\mathbf{0} \in S$ (zero vector).
- (2) If $\mathbf{v}_1, \mathbf{v}_2 \in S$, then $\mathbf{v}_1 + \mathbf{v}_2 \in S$ (vector addition).
- (3) If $\mathbf{v} \in S$ and $\alpha \in \mathbb{R}$, then $\alpha\mathbf{v} \in S$ (scalar multiplication).

If (1), (2) and (3) hold, then S is a subspace of V .

If any of (1), (2) or (3) does not hold, then S is not a subspace of V .

The following worked exercises and exercises illustrate how this strategy is used to show that a given set is a subspace.

Worked Exercise C42

Show that the set of vectors $S = \{(x, 3x) : x \in \mathbb{R}\}$ is a subspace of \mathbb{R}^2 . Sketch this subspace.

Solution

The set S is a subset of \mathbb{R}^2 , so we use Strategy C10.

☁ We first check condition (1): $\mathbf{0} \in S$. ☁

If $x = 0$, then $(x, 3x) = (0, 0)$, so S contains the zero vector of \mathbb{R}^2 .

☁ We check condition (2): If $\mathbf{v}_1, \mathbf{v}_2 \in S$, then $\mathbf{v}_1 + \mathbf{v}_2 \in S$. ☁

Let $\mathbf{v}_1 = (x_1, 3x_1)$ and $\mathbf{v}_2 = (x_2, 3x_2)$ belong to S . Then

$$\begin{aligned}\mathbf{v}_1 + \mathbf{v}_2 &= (x_1, 3x_1) + (x_2, 3x_2) \\ &= (x_1 + x_2, 3x_1 + 3x_2) \\ &= (x_1 + x_2, 3(x_1 + x_2)).\end{aligned}$$

This vector has the correct form for a vector in S , since $x_1 + x_2 \in \mathbb{R}$, so S is closed under vector addition.

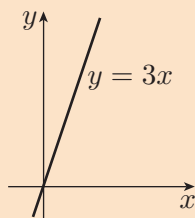
☁ We check condition (3): If $\mathbf{v} \in S$ and $\alpha \in \mathbb{R}$, then $\alpha\mathbf{v} \in S$. ☁

Let $\mathbf{v} = (x, 3x) \in S$ and $\alpha \in \mathbb{R}$. Then

$$\alpha\mathbf{v} = \alpha(x, 3x) = (\alpha x, \alpha 3x) = (\alpha x, 3(\alpha x)).$$

This vector has the correct form for a vector in S , since $\alpha x \in \mathbb{R}$, so S is closed under scalar multiplication.

Since conditions (1), (2) and (3) are satisfied, S is a subspace of \mathbb{R}^2 . This subspace is the line through the origin with equation $y = 3x$.



Exercise C66

Show that the set of vectors $S = \{(x, -2x) : x \in \mathbb{R}\}$ is a subspace of \mathbb{R}^2 .

Worked Exercise C43

Show that the set of vectors $S = \{(x, y, 2x - 3y) : x, y \in \mathbb{R}\}$ is a subspace of \mathbb{R}^3 .

Solution

The set S is a subset of \mathbb{R}^3 , so we use Strategy C10.

If $x = y = 0$, then $(x, y, 2x - 3y) = (0, 0, 0)$, so S contains the zero vector of \mathbb{R}^3 .

Let $\mathbf{v}_1 = (x_1, y_1, 2x_1 - 3y_1)$ and $\mathbf{v}_2 = (x_2, y_2, 2x_2 - 3y_2)$ belong to S . Then

$$\begin{aligned}\mathbf{v}_1 + \mathbf{v}_2 &= (x_1, y_1, 2x_1 - 3y_1) + (x_2, y_2, 2x_2 - 3y_2) \\ &= (x_1 + x_2, y_1 + y_2, 2x_1 - 3y_1 + 2x_2 - 3y_2) \\ &= (x_1 + x_2, y_1 + y_2, 2(x_1 + x_2) - 3(y_1 + y_2)).\end{aligned}$$



This vector has the correct form for a vector in S , since $x_1 + x_2, y_1 + y_2 \in \mathbb{R}$, so S is closed under vector addition.

Let $\mathbf{v} = (x, y, 2x - 3y) \in S$ and $\alpha \in \mathbb{R}$. Then

$$\begin{aligned}\alpha\mathbf{v} &= \alpha(x, y, 2x - 3y) \\ &= (\alpha x, \alpha y, \alpha(2x - 3y)) \\ &= (\alpha x, \alpha y, 2(\alpha x) - 3(\alpha y)).\end{aligned}$$

This vector has the correct form for a vector in S , since $\alpha x, \alpha y \in \mathbb{R}$, so S is closed under scalar multiplication.

Since conditions (1), (2) and (3) are satisfied, S is a subspace of \mathbb{R}^3 .

 S is the set of points in \mathbb{R}^3 satisfying $z = 2x - 3y$; it is the plane through the origin with equation $2x - 3y - z = 0$. 

Strategy C10 is used in much the same way to determine whether a given subset is a subspace. However, since if any one of the conditions fails then the subset is not a subspace, it may be that only one of the conditions needs to be checked.

Worked Exercise C44

For each of the following, determine whether the set S is a subspace of the vector space \mathbb{R}^3 .

- (a) $S = \{(x, y, x - y + 2) : x, y \in \mathbb{R}\}$ (b) $S = \{(z - y, y, z) : y, z \in \mathbb{R}\}$



Solution

In each case the set S is a subset of \mathbb{R}^3 , so we use Strategy C10.

- (a) If $\mathbf{0} \in S$, then $(x, y, x - y + 2) = (0, 0, 0)$ for some numbers x and y . Equating corresponding coordinates, we obtain the system

$$\begin{aligned}x &= 0 \\ y &= 0 \\ x - y &= -2.\end{aligned}$$

This system is inconsistent so has no solution. Therefore $\mathbf{0}$ does not belong to S and condition (1) is not satisfied. Hence S is not a subspace of \mathbb{R}^3 .

 Since condition (1) is not satisfied, we do not need to check conditions (2) and (3). However, neither is satisfied, and either one could have been used to show that S is not a subspace. 

- (b) If $y = z = 0$, then $(z - y, y, z) = (0, 0, 0)$, so S contains the zero vector of \mathbb{R}^3 .

Let $\mathbf{v}_1 = (z_1 - y_1, y_1, z_1)$ and $\mathbf{v}_2 = (z_2 - y_2, y_2, z_2)$ belong to S . Then

$$\begin{aligned}\mathbf{v}_1 + \mathbf{v}_2 &= (z_1 - y_1, y_1, z_1) + (z_2 - y_2, y_2, z_2) \\ &= (z_1 - y_1 + z_2 - y_2, y_1 + y_2, z_1 + z_2) \\ &= ((z_1 + z_2) - (y_1 + y_2), y_1 + y_2, z_1 + z_2).\end{aligned}$$



This vector has the correct form for a vector in S , since $y_1 + y_2, z_1 + z_2 \in \mathbb{R}$, so S is closed under vector addition.

Let $\mathbf{v} = (z - y, y, z) \in S$ and $\alpha \in \mathbb{R}$. Then

$$\begin{aligned}\alpha \mathbf{v} &= \alpha(z - y, y, z) \\ &= (\alpha(z - y), \alpha y, \alpha z) \\ &= (\alpha z - \alpha y, \alpha y, \alpha z).\end{aligned}$$

This vector has the correct form for a vector in S , since $\alpha y, \alpha z \in \mathbb{R}$, so S is closed under scalar multiplication.

Since conditions (1), (2) and (3) are satisfied, S is a subspace of \mathbb{R}^3 .

 S is the set of points in \mathbb{R}^3 satisfying $z = x + y$; it is the plane through the origin with equation $x + y - z = 0$. 

Exercise C67

For each of the following, determine whether the set S is a subspace of the vector space V .

- (a) $V = \mathbb{R}^2$, $S = \{(x, x + 2) : x \in \mathbb{R}\}$.
 (b) $V = \mathbb{R}^4$, $S = \{(x, y, z, x + 2y - z) : x, y, z \in \mathbb{R}\}$.

Worked Exercise C45



Determine whether the set $S = \{a \cos x : a \in \mathbb{R}\}$ is a subspace of the vector space $V = \{a \cos x + b \sin x : a, b \in \mathbb{R}\}$.

(We showed that V is a vector space in Subsection 1.2.)

Solution

The set S is a subset of V , so we use Strategy C10.

The zero vector of V is $0 \cos x + 0 \sin x = 0 = \mathbf{0}$. If $a = 0$, then $a \cos x = 0 \cos x = 0$, so S contains the zero vector.

 We need ‘names’ for two general vectors in S . As they are functions, we call them $f_1(x)$ and $f_2(x)$. 

Let $f_1(x) = a_1 \cos x$ and $f_2(x) = a_2 \cos x$ belong to S . Then

$$f_1(x) + f_2(x) = a_1 \cos x + a_2 \cos x = (a_1 + a_2) \cos x.$$

The function $f_1(x) + f_2(x)$ has the correct form for a vector in S , since $a_1 + a_2 \in \mathbb{R}$, so S is closed under vector addition.

Let $f(x) = a \cos x \in S$ and $\alpha \in \mathbb{R}$. Then

$$\alpha f(x) = \alpha a \cos x = (\alpha a) \cos x.$$

This function has the correct form for a vector in S , since $\alpha a \in \mathbb{R}$, so S is closed under scalar multiplication.

Since conditions (1), (2) and (3) are satisfied, S is a subspace of V .

Exercise C68

For each of the following, determine whether the set S is a subspace of the vector space V .

- (a) $V = P_3$, $S = \{a + bx : a, b \in \mathbb{R}\}$.
- (b) $V = P_3$, $S = \{x + ax^2 : a \in \mathbb{R}\}$.
- (c) $V = M_{2,2}$, $S = \left\{ \begin{pmatrix} a & 1 \\ 0 & d \end{pmatrix} : a, d \in \mathbb{R} \right\}$.

The following theorem shows that the span of a subset of a vector space is always a subspace.

Theorem C28

Let S be a non-empty finite subset of a vector space V . Then $\langle S \rangle$ is a subspace of V .

Proof Let $S = \{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\}$ be a non-empty finite subset of a vector space V . Then the set $\langle S \rangle$ is a subset of V since V is closed under vector addition and scalar multiplication.

 We apply Strategy C10. 

The span $\langle S \rangle$ contains the zero vector, since $0\mathbf{u}_1 + 0\mathbf{u}_2 + \dots + 0\mathbf{u}_n = \mathbf{0}$ belongs to $\langle S \rangle$.

Let $\mathbf{v}_1 = a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + \dots + a_n\mathbf{u}_n$ and $\mathbf{v}_2 = b_1\mathbf{u}_1 + b_2\mathbf{u}_2 + \dots + b_n\mathbf{u}_n$ be any two vectors in $\langle S \rangle$. Then

$$\begin{aligned}\mathbf{v}_1 + \mathbf{v}_2 &= (a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + \dots + a_n\mathbf{u}_n) + (b_1\mathbf{u}_1 + b_2\mathbf{u}_2 + \dots + b_n\mathbf{u}_n) \\ &= (a_1 + b_1)\mathbf{u}_1 + (a_2 + b_2)\mathbf{u}_2 + \dots + (a_n + b_n)\mathbf{u}_n.\end{aligned}$$

This is a member of $\langle S \rangle$, since it is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$. Hence $\langle S \rangle$ is closed under vector addition.

Let $\mathbf{v} = a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + \dots + a_n\mathbf{u}_n$ and $\alpha \in \mathbb{R}$. Then

$$\begin{aligned}\alpha\mathbf{v} &= \alpha(a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + \dots + a_n\mathbf{u}_n) \\ &= (\alpha a_1)\mathbf{u}_1 + (\alpha a_2)\mathbf{u}_2 + \dots + (\alpha a_n)\mathbf{u}_n.\end{aligned}$$

This is a member of $\langle S \rangle$, since it is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$. Hence $\langle S \rangle$ is closed under scalar multiplication.

Thus $\langle S \rangle$ is a subspace of V . ■

4.2 Bases and dimension

In the previous subsection you saw several subspaces of finite-dimensional vector spaces. Since these subspaces are all vector spaces in their own right, they have bases and dimensions, and we look at these in this subsection.

Let us return to two of our earlier examples from Section 2: Worked Exercises C32(a) and (b).

By Theorem C28, we now know that the set of vectors in \mathbb{R}^2 spanned by the set $S = \{(1, 1)\}$ is a subspace of \mathbb{R}^2 . In Worked Exercise C32(a) we saw that any vector in this subspace $\langle S \rangle$ can be written in the form (α, α) for some $\alpha \in \mathbb{R}$; so the set $\{(1, 1)\}$ is a basis for this subspace. Thus the dimension of the subspace is 1. This agrees with our intuitive idea of dimension: we saw that these vectors form a line through the origin – the line $y = x$, as shown in Figure 11 – which is one-dimensional.

Similarly, from Worked Exercise C32(b) the set of vectors in \mathbb{R}^3 spanned by the set $S = \{(1, 0, 1), (2, 0, 3)\}$ is a subspace of \mathbb{R}^3 . This subspace $\langle S \rangle$ consists of those points of \mathbb{R}^3 of the form $(x, 0, z)$. Since the set $\{(1, 0, 1), (2, 0, 3)\}$ spans the subspace and is linearly independent (the vectors are not multiples of each other), it is a basis for this subspace. Since there are two vectors in the basis, the dimension of the subspace is 2.

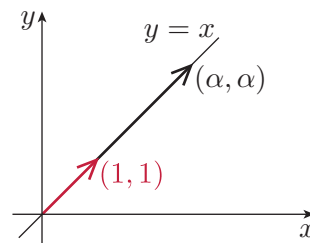


Figure 11 The one-dimensional subspace $\langle \{(1, 1)\} \rangle$

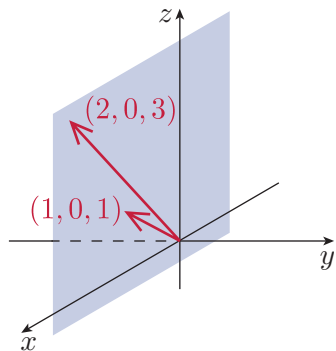


Figure 12 The two-dimensional subspace $\langle \{(1, 0, 1), (2, 0, 3)\} \rangle$

Again, this links the idea of dimension in linear algebra to our intuitive idea of dimension: we saw that the subspace spanned by these two vectors is a plane through the origin – namely, the plane $y = 0$, as shown in Figure 12 – which is two-dimensional. Since any vector in the subspace can be written in the form $(x, 0, z)$, we can find another basis for this subspace by writing

$$(x, 0, z) = x(1, 0, 0) + z(0, 0, 1).$$

This means that the set $\{(1, 0, 0), (0, 0, 1)\}$ is another spanning set for the subspace and, as it is also linearly independent, it is a basis for the subspace. This basis has the additional advantage that it is *orthogonal*, which means that the basis vectors are at right angles to each other. We will return to orthogonal bases in Section 5.

In the following worked exercises and exercises we consider various subspaces of \mathbb{R}^3 and \mathbb{R}^4 and look at their bases and dimension.

Worked Exercise C46

Find the equation of the subspace of \mathbb{R}^3 spanned by the set $\{(1, 0, 2), (2, 3, 4)\}$.

Solution

The two vectors are not multiples of each other, so they are linearly independent.

Since $\{(1, 0, 2), (2, 3, 4)\}$ is a linearly independent set, the subspace it spans is a two-dimensional subspace of \mathbb{R}^3 (by Theorem C25).

A two-dimensional subspace is a plane, and since the zero vector is in the subspace this plane must pass through the origin.

The subspace is therefore a plane through the origin with equation

$$ax + by + cz = 0,$$

where a, b, c are not all zero.

Since the vectors in the spanning set lie in the plane, the values of a, b and c must satisfy the system

$$\begin{aligned} a + 2c &= 0 \\ 2a + 3b + 4c &= 0. \end{aligned}$$

The first of these equations gives $a = -2c$, and substituting this into the second equation gives $b = 0$; so the subspace is the plane with equation $-2cx + cz = 0$, or, equivalently,

$$2x - z = 0.$$

Exercise C69



Find the equation of the subspace of \mathbb{R}^3 spanned by the set $\{(1, -2, 0), (0, 3, 3)\}$.

Worked Exercise C47

Find a basis for the subspace $S = \{(z - y, y, z) : y, z \in \mathbb{R}\}$ of \mathbb{R}^3 , and hence write down the dimension of S .

(You showed that S is a subspace of \mathbb{R}^3 in Worked Exercise C44(b).)

Solution

 We use the form of the vectors in S to help us find a possible basis: the coordinates of the general vector involve y and z , so we look for vectors \mathbf{v}_1 and \mathbf{v}_2 such that $(z - y, y, z) = z\mathbf{v}_1 + y\mathbf{v}_2$. 

Since

$$\begin{aligned}(z - y, y, z) &= (z, 0, z) + (-y, y, 0) \\ &= z(1, 0, 1) + y(-1, 1, 0),\end{aligned}$$

any vector in S can be written as a linear combination of the vectors in the set $\{(1, 0, 1), (-1, 1, 0)\}$, so this set spans S .

The vectors in the set are also linearly independent, as they are not multiples of each other, so $\{(1, 0, 1), (-1, 1, 0)\}$ is a basis for S .

Therefore S has dimension 2.

 S is the set of points of \mathbb{R}^3 satisfying $x = z - y$; it is the plane with equation $x + y - z = 0$. 

Exercise C70

Find a basis for the subspace

$$S = \{(x, y, z, x + 2y - z) : x, y, z \in \mathbb{R}\}$$

of \mathbb{R}^4 , and hence write down the dimension of S .



(You showed that S is a subspace of \mathbb{R}^4 in Exercise C67(b).)

Worked Exercise C48

Find a basis for the plane $x - 3y + 2z = 0$ (a subspace of \mathbb{R}^3).

Solution

Since the subspace is a plane, it has dimension 2, and so has two basis vectors.

 We need to find two vectors that lie in the plane and form a linearly independent set. There are many choices, so we first set $x = 0$ and then set $z = 0$ to find two vectors in the plane. 


The vectors $(0, 2, 3)$ and $(3, 1, 0)$ both lie in the plane, and they are linearly independent, since one is not a multiple of the other.

Therefore $\{(0, 2, 3), (3, 1, 0)\}$ is a basis for the subspace $x - 3y + 2z = 0$.

The following result, which will be used in Unit C3, has been illustrated by the worked exercises and exercises in this subsection. For example, in Worked Exercise C47 we had $V = \mathbb{R}^3$, so $\dim V = 3$ and $\dim S = 2 \leq \dim V$.

Theorem C29

The dimension of a subspace of a vector space V is less than or equal to the dimension of V .

Proof Let V be a vector space of dimension n , and let S be a subspace of V . Suppose that the dimension of S is m , and let $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_m\}$ be a basis for S . Then $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_m\}$ is a linearly independent set of vectors in V . Thus $m \leq n$ by Corollary C23. 

5 Orthogonal bases

In this section you will look at bases in which the basis vectors are all *orthogonal* to each other.

5.1 Orthogonal bases in \mathbb{R}^3

Suppose that we wish to express the vector $(10, 0, 4)$ in \mathbb{R}^3 in terms of the basis

$$\{(2, 1, 1), (1, -4, 2), (-2, 1, 3)\}.$$

Using the method given in Subsection 2.1, we first write

$$(10, 0, 4) = \alpha_1(2, 1, 1) + \alpha_2(1, -4, 2) + \alpha_3(-2, 1, 3).$$

Equating corresponding coordinates gives the system

$$\begin{aligned} 2\alpha_1 + \alpha_2 - 2\alpha_3 &= 10 \\ \alpha_1 - 4\alpha_2 + \alpha_3 &= 0 \\ \alpha_1 + 2\alpha_2 + 3\alpha_3 &= 4. \end{aligned}$$

We can solve this system using Gauss–Jordan elimination or directly, to obtain the solution

$$\alpha_1 = 4, \quad \alpha_2 = \frac{6}{7}, \quad \alpha_3 = -\frac{4}{7}.$$

Thus

$$(10, 0, 4) = 4(2, 1, 1) + \frac{6}{7}(1, -4, 2) - \frac{4}{7}(-2, 1, 3).$$

In this section you will see that there is a simpler method than this that involves scalar products of vectors. It can be used when, as here, the given basis is an *orthogonal* basis. In this subsection we concentrate on \mathbb{R}^3 .

We start by recalling from Unit A1 the definition of the scalar product in \mathbb{R}^3 , and then use this to define the term *orthogonal*.

Definitions

Let $\mathbf{v}_1 = (x_1, y_1, z_1)$ and $\mathbf{v}_2 = (x_2, y_2, z_2)$ be vectors in \mathbb{R}^3 .

The **scalar product** of \mathbf{v}_1 and \mathbf{v}_2 is the real number

$$\mathbf{v}_1 \cdot \mathbf{v}_2 = x_1x_2 + y_1y_2 + z_1z_2.$$

The vectors \mathbf{v}_1 and \mathbf{v}_2 in \mathbb{R}^3 are **orthogonal** if $\mathbf{v}_1 \cdot \mathbf{v}_2 = 0$.

For example, the vectors $\mathbf{v}_1 = (2, 1, 1)$ and $\mathbf{v}_2 = (-2, 1, 3)$ are orthogonal, since

$$\mathbf{v}_1 \cdot \mathbf{v}_2 = 2 \times (-2) + 1 \times 1 + 1 \times 3 = -4 + 1 + 3 = 0.$$

Geometrically, this means that the vectors \mathbf{v}_1 and \mathbf{v}_2 are at right angles to each other, as shown in Figure 13.

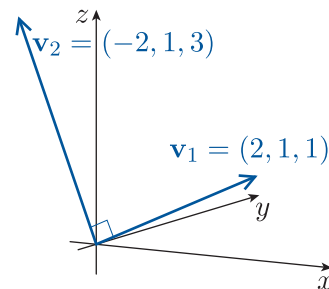


Figure 13 The orthogonal vectors $\mathbf{v}_1 = (2, 1, 1)$ and $\mathbf{v}_2 = (-2, 1, 3)$

Exercise C71

- Show that $(2, 1, 1)$ and $(1, -4, 2)$ are orthogonal.
- Determine which pairs of the following vectors are orthogonal:

$$\mathbf{v}_1 = (-2, 6, 1), \quad \mathbf{v}_2 = (9, 2, 6), \quad \mathbf{v}_3 = (4, -15, -1).$$

Definition

A set of vectors in \mathbb{R}^3 is an **orthogonal set** if every pair of distinct vectors in the set is orthogonal.

For example, $\{\mathbf{v}_1, \mathbf{v}_2\}$ is an orthogonal set if $\mathbf{v}_1 \cdot \mathbf{v}_2 = 0$; we have therefore shown above that $\{(2, 1, 1), (1, -4, 2)\}$ is an orthogonal set.

Similarly, $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is an orthogonal set if

$$\mathbf{v}_1 \cdot \mathbf{v}_2 = \mathbf{v}_1 \cdot \mathbf{v}_3 = \mathbf{v}_2 \cdot \mathbf{v}_3 = 0.$$

So $\{(2, 1, 1), (1, -4, 2), (-2, 1, 3)\}$ is an orthogonal set since



$$(1, -4, 2) \cdot (-2, 1, 3) = -2 - 4 + 6 = 0,$$

and we have shown that $(2, 1, 1) \cdot (1, -4, 2) = 0$ and $(2, 1, 1) \cdot (-2, 1, 3) = 0$.

One of the most useful features of orthogonal sets of non-zero vectors is their linear independence. The following proof is for sets of three non-zero vectors, but a similar proof applies to other numbers of vectors and indeed to orthogonal sets of vectors in \mathbb{R}^n .

Theorem C30

Let $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ be an orthogonal set of non-zero vectors in \mathbb{R}^3 . Then \mathbf{v}_1 , \mathbf{v}_2 and \mathbf{v}_3 are linearly independent.

Proof  To show that \mathbf{v}_1 , \mathbf{v}_2 and \mathbf{v}_3 are linearly independent we need to deduce that if $\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \alpha_3\mathbf{v}_3 = \mathbf{0}$ then $\alpha_1 = \alpha_2 = \alpha_3 = 0$ by using the properties of scalar products. 

Suppose that

$$\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \alpha_3\mathbf{v}_3 = \mathbf{0}.$$

We form the scalar product on both sides of the equation with \mathbf{v}_1 :

$$\mathbf{v}_1 \cdot (\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \alpha_3\mathbf{v}_3) = \mathbf{v}_1 \cdot \mathbf{0} = 0.$$

Using the multiples property of the scalar product (Unit A1) we get

$$\alpha_1(\mathbf{v}_1 \cdot \mathbf{v}_1) + \alpha_2(\mathbf{v}_1 \cdot \mathbf{v}_2) + \alpha_3(\mathbf{v}_1 \cdot \mathbf{v}_3) = 0.$$

Since $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is an orthogonal set of non-zero vectors in \mathbb{R}^3 , we know that

$$\mathbf{v}_1 \cdot \mathbf{v}_1 \neq 0, \mathbf{v}_1 \cdot \mathbf{v}_2 = 0, \mathbf{v}_1 \cdot \mathbf{v}_3 = 0,$$

so we have $\alpha_1(\mathbf{v}_1 \cdot \mathbf{v}_1) = 0$ and thus $\alpha_1 = 0$.

Similarly, we form the scalar product with \mathbf{v}_2 and \mathbf{v}_3 :

$$\mathbf{v}_2 \cdot (\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \alpha_3\mathbf{v}_3) = \mathbf{v}_2 \cdot \mathbf{0} = 0,$$

which gives $\alpha_2 = 0$;

$$\mathbf{v}_3 \cdot (\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \alpha_3\mathbf{v}_3) = \mathbf{v}_3 \cdot \mathbf{0} = 0,$$

which gives $\alpha_3 = 0$.

We conclude that if $\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \alpha_3\mathbf{v}_3 = \mathbf{0}$ then $\alpha_1 = \alpha_2 = \alpha_3 = 0$.

Thus $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a linearly independent set. 

This result leads to the idea of an *orthogonal basis*.

You have seen that any linearly independent set of three vectors in \mathbb{R}^3 is a basis for \mathbb{R}^3 . Now, if we have an orthogonal set of three non-zero vectors in \mathbb{R}^3 , then we know from Theorem C30 that the set is linearly independent, so the set is a basis for \mathbb{R}^3 . We call an orthogonal set that is a basis an **orthogonal basis**.

Theorem C31

Any orthogonal set of three non-zero vectors in \mathbb{R}^3 is an orthogonal basis for \mathbb{R}^3 .

For example, the standard basis $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ for \mathbb{R}^3 is an orthogonal basis, because these three basis vectors form an orthogonal set. Similarly, the triple of vectors below is an orthogonal basis for \mathbb{R}^3 since the vectors are orthogonal (as we saw above), there are three of them, and they are all non-zero:

$$\{(2, 1, 1), (1, -4, 2), (-2, 1, 3)\}.$$

One reason that orthogonal bases are so important is that it is usually much easier to express a vector in terms of an orthogonal basis than in terms of a general basis. At the beginning of this subsection we expressed $(10, 0, 4)$ in terms of the orthogonal basis $\{(2, 1, 1), (1, -4, 2), (-2, 1, 3)\}$ by writing

$$(10, 0, 4) = \alpha_1(2, 1, 1) + \alpha_2(1, -4, 2) + \alpha_3(-2, 1, 3) \quad (3)$$

and solving the resulting system of linear equations.

However, there is a quicker way of solving equation (3) because the basis is an orthogonal basis. We take the scalar product of the vector $(10, 0, 4)$ expressed as in equation (3) with each basis vector in turn, making use of the fact that the scalar product of orthogonal vectors is zero.

First with $(2, 1, 1)$:

$$\begin{aligned} (2, 1, 1) \cdot (10, 0, 4) &= \alpha_1(2, 1, 1) \cdot (2, 1, 1) + \alpha_2(2, 1, 1) \cdot (1, -4, 2) \\ &\quad + \alpha_3(2, 1, 1) \cdot (-2, 1, 3) \\ &= \alpha_1(2, 1, 1) \cdot (2, 1, 1) + 0 + 0. \end{aligned}$$

The equation above gives

$$\alpha_1 = \frac{(2, 1, 1) \cdot (10, 0, 4)}{(2, 1, 1) \cdot (2, 1, 1)} = \frac{24}{6} = 4.$$

Similarly, taking the scalar product with $(1, -4, 2)$:

$$(1, -4, 2) \cdot (10, 0, 4) = 0 + \alpha_2(1, -4, 2) \cdot (1, -4, 2) + 0.$$

Thus

$$\alpha_2 = \frac{(1, -4, 2) \cdot (10, 0, 4)}{(1, -4, 2) \cdot (1, -4, 2)} = \frac{18}{21} = \frac{6}{7}.$$

Finally, taking the scalar product with $(-2, 1, 3)$:

$$(-2, 1, 3) \cdot (10, 0, 4) = 0 + 0 + \alpha_3(-2, 1, 3) \cdot (-2, 1, 3).$$

Thus

$$\alpha_3 = \frac{(-2, 1, 3) \cdot (10, 0, 4)}{(-2, 1, 3) \cdot (-2, 1, 3)} = \frac{-8}{14} = -\frac{4}{7}.$$

Therefore, we have $\alpha_1 = 4$, $\alpha_2 = \frac{6}{7}$ and $\alpha_3 = -\frac{4}{7}$, so

$$(10, 0, 4) = 4(2, 1, 1) + \frac{6}{7}(1, -4, 2) - \frac{4}{7}(-2, 1, 3).$$

This procedure works for orthogonal bases in general in \mathbb{R}^3 and is summarised in the following strategy.

Strategy C11

To express a vector \mathbf{u} in \mathbb{R}^3 in terms of an orthogonal basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$:

1. calculate $\alpha_1 = \frac{\mathbf{v}_1 \cdot \mathbf{u}}{\mathbf{v}_1 \cdot \mathbf{v}_1}$, $\alpha_2 = \frac{\mathbf{v}_2 \cdot \mathbf{u}}{\mathbf{v}_2 \cdot \mathbf{v}_2}$ and $\alpha_3 = \frac{\mathbf{v}_3 \cdot \mathbf{u}}{\mathbf{v}_3 \cdot \mathbf{v}_3}$
2. write $\mathbf{u} = \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \alpha_3\mathbf{v}_3$.

Exercise C72

- (a) Verify that $\{(3, 4, 0), (8, -6, 0), (0, 0, 5)\}$ is an orthogonal basis for \mathbb{R}^3 .
- (b) Express the vector $(10, 0, 4)$ in terms of this basis.

5.2 Orthogonal bases in \mathbb{R}^n

In this subsection we see how the definitions and results of the previous subsection can be generalised to \mathbb{R}^n , for any positive integer n . We start with the definition of the scalar product of vectors.

Definition

Let $\mathbf{v} = (v_1, v_2, \dots, v_n)$ and $\mathbf{w} = (w_1, w_2, \dots, w_n)$ be vectors in \mathbb{R}^n . The **scalar product** of \mathbf{v} and \mathbf{w} is the real number

$$\mathbf{v} \cdot \mathbf{w} = v_1w_1 + v_2w_2 + \dots + v_nw_n.$$

For example, in \mathbb{R}^5 the scalar product of the vectors $\mathbf{v} = (1, 2, 3, 4, 5)$ and $\mathbf{w} = (3, -4, 0, 3, -2)$ is

$$\begin{aligned} \mathbf{v} \cdot \mathbf{w} &= 1 \times 3 + 2 \times (-4) + 3 \times 0 + 4 \times 3 + 5 \times (-2) \\ &= 3 - 8 + 0 + 12 - 10 = -3. \end{aligned}$$

Exercise C73

Calculate the following scalar products.

- (a) $(1, 2, -1, 0) \cdot (0, -5, 6, -3)$ in \mathbb{R}^4 .
 (b) $(1, 2, 3, 4, 5, 6) \cdot (3, 2, 1, 0, -1, -2)$ in \mathbb{R}^6 .

We now see how the ideas of an orthogonal set and an orthogonal basis extend to \mathbb{R}^n .

Definitions

The vectors \mathbf{v} and \mathbf{w} in \mathbb{R}^n are **orthogonal** if $\mathbf{v} \cdot \mathbf{w} = 0$.

A set of vectors in \mathbb{R}^n is an **orthogonal set** if every pair of distinct vectors in the set is orthogonal.

An **orthogonal basis** for \mathbb{R}^n is an orthogonal set that is a basis for \mathbb{R}^n .

For example, in \mathbb{R}^6 the set

$$\{(1, 1, 1, 1, 1, 1), (2, -2, 2, -2, 2, -2), (5, 5, 0, 0, -5, -5)\}$$

is an orthogonal set, since

$$\begin{aligned} (1, 1, 1, 1, 1, 1) \cdot (2, -2, 2, -2, 2, -2) \\ = 2 - 2 + 2 - 2 + 2 - 2 = 0, \end{aligned}$$

$$\begin{aligned} (1, 1, 1, 1, 1, 1) \cdot (5, 5, 0, 0, -5, -5) \\ = 5 + 5 + 0 + 0 - 5 - 5 = 0 \end{aligned}$$

and

$$\begin{aligned} (2, -2, 2, -2, 2, -2) \cdot (5, 5, 0, 0, -5, -5) \\ = 10 - 10 + 0 + 0 - 10 + 10 = 0. \end{aligned}$$

Exercise C74

Show that the set $\{(1, 0, 0, 0, 0), (0, 2, 0, 0, 0), (0, 0, 1, 1, 0)\}$ is an orthogonal set in \mathbb{R}^5 .

Note that the standard basis

$$\{(1, 0, \dots, 0), (0, 1, 0, \dots, 0), \dots, (0, \dots, 0, 1)\}$$

is an orthogonal basis for \mathbb{R}^n .

In Subsection 5.1 you saw that any orthogonal set of three non-zero vectors in \mathbb{R}^3 is linearly independent and therefore forms an orthogonal basis for \mathbb{R}^3 . Exactly the same methods can be used to prove the following more general result.

Theorem C32

Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ be an orthogonal set of non-zero vectors in \mathbb{R}^n . Then S is a linearly independent set.

Since any set of n linearly independent vectors in \mathbb{R}^n forms a basis for \mathbb{R}^n , we obtain the following corollary to Theorem C32.

Corollary C33

Any orthogonal set of n non-zero vectors in \mathbb{R}^n is an orthogonal basis for \mathbb{R}^n .

Exercise C75

Show that

$$\{(1, 2, 1, 0), (-1, 1, -1, 1), (1, 0, -1, 0), (1, -1, 1, 3)\}$$

is an orthogonal basis for \mathbb{R}^4 .

Expressing vectors in terms of orthogonal bases

Given an orthogonal basis for \mathbb{R}^n , it is particularly easy to express any given vector as a linear combination of the basis vectors. As for \mathbb{R}^3 in Subsection 5.1, we simply need to calculate scalar products: we do not need to solve a system of linear equations.

Theorem C34

Let $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ be an orthogonal basis for \mathbb{R}^n and let \mathbf{u} be any vector in \mathbb{R}^n . Then

$$\mathbf{u} = \left(\frac{\mathbf{v}_1 \cdot \mathbf{u}}{\mathbf{v}_1 \cdot \mathbf{v}_1} \right) \mathbf{v}_1 + \left(\frac{\mathbf{v}_2 \cdot \mathbf{u}}{\mathbf{v}_2 \cdot \mathbf{v}_2} \right) \mathbf{v}_2 + \cdots + \left(\frac{\mathbf{v}_n \cdot \mathbf{u}}{\mathbf{v}_n \cdot \mathbf{v}_n} \right) \mathbf{v}_n.$$

Proof Let $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ be an orthogonal basis for \mathbb{R}^n and let \mathbf{u} be any vector in \mathbb{R}^n . Since $\mathbf{u} \in \mathbb{R}^n$, we can write \mathbf{u} as a linear combination of the basis vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$:

$$\mathbf{u} = \alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \dots + \alpha_n \mathbf{v}_n. \quad (4)$$

Forming the scalar product of both sides of equation (4) with \mathbf{v}_1 gives

$$\mathbf{v}_1 \cdot \mathbf{u} = \alpha_1 (\mathbf{v}_1 \cdot \mathbf{v}_1) \quad (\text{all other terms are } 0),$$

$$\text{so } \alpha_1 = \frac{\mathbf{v}_1 \cdot \mathbf{u}}{\mathbf{v}_1 \cdot \mathbf{v}_1}.$$

Similarly, forming the scalar product of both sides of equation (4) with \mathbf{v}_2 gives

$$\mathbf{v}_2 \cdot \mathbf{u} = \alpha_2 (\mathbf{v}_2 \cdot \mathbf{v}_2) \quad (\text{all other terms are } 0),$$

$$\text{so } \alpha_2 = \frac{\mathbf{v}_2 \cdot \mathbf{u}}{\mathbf{v}_2 \cdot \mathbf{v}_2}.$$

Continuing in this way, we deduce that

$$\alpha_i = \frac{\mathbf{v}_i \cdot \mathbf{u}}{\mathbf{v}_i \cdot \mathbf{v}_i} \quad \text{for each } i = 1, 2, \dots, n.$$

Thus

$$\mathbf{u} = \left(\frac{\mathbf{v}_1 \cdot \mathbf{u}}{\mathbf{v}_1 \cdot \mathbf{v}_1} \right) \mathbf{v}_1 + \left(\frac{\mathbf{v}_2 \cdot \mathbf{u}}{\mathbf{v}_2 \cdot \mathbf{v}_2} \right) \mathbf{v}_2 + \dots + \left(\frac{\mathbf{v}_n \cdot \mathbf{u}}{\mathbf{v}_n \cdot \mathbf{v}_n} \right) \mathbf{v}_n,$$

as required. ■

The result of Theorem C34 can be expressed in the form of a strategy that generalises Strategy C11.

Strategy C12

To express a vector \mathbf{u} in \mathbb{R}^n in terms of an orthogonal basis $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$:

1. calculate $\alpha_1 = \frac{\mathbf{v}_1 \cdot \mathbf{u}}{\mathbf{v}_1 \cdot \mathbf{v}_1}$, $\alpha_2 = \frac{\mathbf{v}_2 \cdot \mathbf{u}}{\mathbf{v}_2 \cdot \mathbf{v}_2}$, \dots , $\alpha_n = \frac{\mathbf{v}_n \cdot \mathbf{u}}{\mathbf{v}_n \cdot \mathbf{v}_n}$
2. write $\mathbf{u} = \alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \dots + \alpha_n \mathbf{v}_n$.

Exercise C76

Express the vector $(1, 2, 3, 4)$ in terms of the orthogonal basis for \mathbb{R}^4 $\{(1, 2, 1, 0), (-1, 1, -1, 1), (1, 0, -1, 0), (1, -1, 1, 3)\}$.

(You showed that this basis is orthogonal in Exercise C75.)

5.3 Constructing orthogonal bases

We now consider how to find an orthogonal basis.

Suppose we want to find an orthogonal basis for \mathbb{R}^3 containing the vector $(2, 1, 1)$. This means that we need to find two more vectors orthogonal to each other and orthogonal to the vector $(2, 1, 1)$.

Now recall from Unit A1 that in \mathbb{R}^3 a vector normal to a plane is perpendicular (orthogonal) to every vector in this plane. Thus to find such a pair of vectors, we can find two orthogonal vectors in the plane through the origin that has normal vector $(2, 1, 1)$.

Using the vector equation of a plane from Unit A1, the vector equation of a plane through the origin with normal vector \mathbf{n} is

$$\mathbf{x} \cdot \mathbf{n} = 0,$$

so here we have $(x, y, z) \cdot (2, 1, 1) = 0$; that is, the equation of the plane is

$$2x + y + z = 0.$$

Rather than pulling two *orthogonal* vectors \mathbf{v}_1 and \mathbf{v}_2 in this plane out of a hat, we start with *any* pair of linearly independent vectors in this plane and follow a method known as the *Gram–Schmidt orthogonalisation process* to construct a pair of orthogonal vectors.



Erhard Schmidt



Jørgen Pedersen Gram

In 1907, the German mathematician Erhard Schmidt (1876–1959) published an orthogonalisation algorithm, which became widely used. Schmidt acknowledged that his process was essentially the same as that published by the Danish mathematician Jørgen Pedersen Gram (1850–1916) in 1883. It appears that their names were first linked together in the 1930s. A related algorithm (now known as *modified Gram–Schmidt*) had been used much earlier by the French mathematician and scientist Pierre-Simon Laplace (1749–1827) in an attempt to estimate the masses of Jupiter and Saturn using the astronomical data of six planets.

To find a pair of linearly independent vectors in the plane $2x + y + z = 0$, we need to find any two vectors in this plane that are not multiples of one another. We choose suitable vectors that are as simple as possible, for example, ones containing small numbers and zeros. We start by setting x to 1 and then setting z and y to 0 in turn, to get a pair of vectors. This gives

$$\mathbf{w}_1 = (1, -2, 0) \quad \text{and} \quad \mathbf{w}_2 = (1, 0, -2).$$

Since these vectors are linearly independent, the set $\{\mathbf{w}_1, \mathbf{w}_2\}$ forms a basis for this plane. (Any other pair of linearly independent vectors in the plane would do just as well.)

We take the first vector \mathbf{v}_1 in our orthogonal basis to be the first of these vectors, so

$$\mathbf{v}_1 = \mathbf{w}_1 = (1, -2, 0).$$

For the second vector \mathbf{v}_2 in our orthogonal basis, we start with \mathbf{w}_2 and then subtract from it a suitable multiple α of \mathbf{v}_1 , chosen so that \mathbf{v}_1 and \mathbf{v}_2 are orthogonal, as illustrated in Figure 14. Since \mathbf{v}_2 is a linear combination of vectors in the plane and the plane is a subspace, we know that \mathbf{v}_2 is also in the plane.

So we set

$$\mathbf{v}_2 = \mathbf{w}_2 - \alpha \mathbf{v}_1;$$

that is,

$$\mathbf{v}_2 = (1, 0, -2) - \alpha(1, -2, 0).$$

We want to find the value of α so that \mathbf{v}_1 and \mathbf{v}_2 are orthogonal.

Therefore we must have

$$\begin{aligned} \mathbf{v}_1 \cdot \mathbf{v}_2 &= \mathbf{v}_1 \cdot (\mathbf{w}_2 - \alpha \mathbf{v}_1) \\ &= \mathbf{v}_1 \cdot \mathbf{w}_2 - \alpha \mathbf{v}_1 \cdot \mathbf{v}_1 \\ &= 0. \end{aligned}$$

Hence

$$\alpha = \frac{\mathbf{v}_1 \cdot \mathbf{w}_2}{\mathbf{v}_1 \cdot \mathbf{v}_1};$$

that is, in this case

$$\alpha = \frac{(1, -2, 0) \cdot (1, 0, -2)}{(1, -2, 0) \cdot (1, -2, 0)} = \frac{1}{5}.$$

Thus

$$\mathbf{v}_2 = (1, 0, -2) - \frac{1}{5}(1, -2, 0) = \left(\frac{4}{5}, \frac{2}{5}, -2\right).$$

So an orthogonal basis for the plane is $\left\{(1, -2, 0), \left(\frac{4}{5}, \frac{2}{5}, -2\right)\right\}$.

Returning to the original problem, this means that we have found that an orthogonal basis for \mathbb{R}^3 containing the vector $(2, 1, 1)$ is

$$\left\{(2, 1, 1), (1, -2, 0), \left(\frac{4}{5}, \frac{2}{5}, -2\right)\right\}.$$

The next exercise asks you to find an orthogonal basis for \mathbb{R}^3 containing a given vector by using the above method.

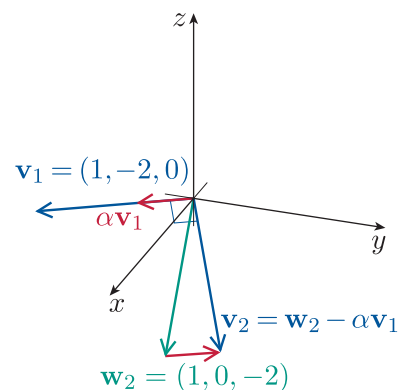


Figure 14 Subtracting a bit of \mathbf{v}_1 from \mathbf{w}_2 to get an orthogonal vector

Exercise C77

- (a) Find the equation of the plane through the origin with normal vector $\mathbf{n} = (3, -4, 5)$.
- (b) Show that the vectors $\mathbf{w}_1 = (4, 3, 0)$ and $\mathbf{w}_2 = (0, 5, 4)$ lie in this plane.
- (c) Find an orthogonal basis $\{\mathbf{v}_1, \mathbf{v}_2\}$ for the plane where $\mathbf{v}_1 = \mathbf{w}_1$, and
- $$\mathbf{v}_2 = \mathbf{w}_2 - \frac{\mathbf{v}_1 \cdot \mathbf{w}_2}{\mathbf{v}_1 \cdot \mathbf{v}_1} \mathbf{v}_1.$$
- (d) Hence write down an orthogonal basis for \mathbb{R}^3 containing the vector $(3, -4, 5)$.

In these examples we started with a pair of arbitrary basis vectors for a plane and adjusted the second to obtain a pair of orthogonal basis vectors. This method can be extended to higher-dimensional spaces by starting with an arbitrary basis and adjusting the basis vectors one by one to obtain an orthogonal basis. It is called the *Gram–Schmidt orthogonalisation process*.

Theorem C35 Gram–Schmidt orthogonalisation process

Let $\{\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_n\}$ be a basis for \mathbb{R}^n , and let

$$\mathbf{v}_1 = \mathbf{w}_1,$$



$$\mathbf{v}_2 = \mathbf{w}_2 - \left(\frac{\mathbf{v}_1 \cdot \mathbf{w}_2}{\mathbf{v}_1 \cdot \mathbf{v}_1} \right) \mathbf{v}_1,$$

$$\mathbf{v}_3 = \mathbf{w}_3 - \left(\frac{\mathbf{v}_1 \cdot \mathbf{w}_3}{\mathbf{v}_1 \cdot \mathbf{v}_1} \right) \mathbf{v}_1 - \left(\frac{\mathbf{v}_2 \cdot \mathbf{w}_3}{\mathbf{v}_2 \cdot \mathbf{v}_2} \right) \mathbf{v}_2,$$

$$\vdots$$

$$\begin{aligned} \mathbf{v}_n = & \mathbf{w}_n - \left(\frac{\mathbf{v}_1 \cdot \mathbf{w}_n}{\mathbf{v}_1 \cdot \mathbf{v}_1} \right) \mathbf{v}_1 - \left(\frac{\mathbf{v}_2 \cdot \mathbf{w}_n}{\mathbf{v}_2 \cdot \mathbf{v}_2} \right) \mathbf{v}_2 \\ & - \dots - \left(\frac{\mathbf{v}_{n-1} \cdot \mathbf{w}_n}{\mathbf{v}_{n-1} \cdot \mathbf{v}_{n-1}} \right) \mathbf{v}_{n-1}. \end{aligned}$$

Then $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is an orthogonal basis for \mathbb{R}^n .

Proof  We show that each vector in the set $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is orthogonal to every other vector in the set. 

We note first that \mathbf{v}_2 is orthogonal to \mathbf{v}_1 , since

$$\begin{aligned} \mathbf{v}_1 \cdot \mathbf{v}_2 &= \mathbf{v}_1 \cdot \left(\mathbf{w}_2 - \left(\frac{\mathbf{v}_1 \cdot \mathbf{w}_2}{\mathbf{v}_1 \cdot \mathbf{v}_1} \right) \mathbf{v}_1 \right) \\ &= (\mathbf{v}_1 \cdot \mathbf{w}_2) - \left(\frac{\mathbf{v}_1 \cdot \mathbf{w}_2}{\mathbf{v}_1 \cdot \mathbf{v}_1} \right) (\mathbf{v}_1 \cdot \mathbf{v}_1) \\ &= (\mathbf{v}_1 \cdot \mathbf{w}_2) - (\mathbf{v}_1 \cdot \mathbf{w}_2) = 0. \end{aligned}$$

Next we note that \mathbf{v}_3 is orthogonal to both \mathbf{v}_1 and \mathbf{v}_2 , since

$$\begin{aligned}\mathbf{v}_1 \cdot \mathbf{v}_3 &= \mathbf{v}_1 \cdot \left(\mathbf{w}_3 - \left(\frac{\mathbf{v}_1 \cdot \mathbf{w}_3}{\mathbf{v}_1 \cdot \mathbf{v}_1} \right) \mathbf{v}_1 - \left(\frac{\mathbf{v}_2 \cdot \mathbf{w}_3}{\mathbf{v}_2 \cdot \mathbf{v}_2} \right) \mathbf{v}_2 \right) \\ &= (\mathbf{v}_1 \cdot \mathbf{w}_3) - \left(\frac{\mathbf{v}_1 \cdot \mathbf{w}_3}{\mathbf{v}_1 \cdot \mathbf{v}_1} \right) (\mathbf{v}_1 \cdot \mathbf{v}_1) - \left(\frac{\mathbf{v}_2 \cdot \mathbf{w}_3}{\mathbf{v}_2 \cdot \mathbf{v}_2} \right) (\mathbf{v}_1 \cdot \mathbf{v}_2) \\ &= (\mathbf{v}_1 \cdot \mathbf{w}_3) - (\mathbf{v}_1 \cdot \mathbf{w}_3) - 0 \\ &= 0\end{aligned}$$

because \mathbf{v}_1 and \mathbf{v}_2 are orthogonal.

Similarly,

$$\begin{aligned}\mathbf{v}_2 \cdot \mathbf{v}_3 &= \mathbf{v}_2 \cdot \left(\mathbf{w}_3 - \left(\frac{\mathbf{v}_1 \cdot \mathbf{w}_3}{\mathbf{v}_1 \cdot \mathbf{v}_1} \right) \mathbf{v}_1 - \left(\frac{\mathbf{v}_2 \cdot \mathbf{w}_3}{\mathbf{v}_2 \cdot \mathbf{v}_2} \right) \mathbf{v}_2 \right) \\ &= (\mathbf{v}_2 \cdot \mathbf{w}_3) - \left(\frac{\mathbf{v}_1 \cdot \mathbf{w}_3}{\mathbf{v}_1 \cdot \mathbf{v}_1} \right) (\mathbf{v}_2 \cdot \mathbf{v}_1) - \left(\frac{\mathbf{v}_2 \cdot \mathbf{w}_3}{\mathbf{v}_2 \cdot \mathbf{v}_2} \right) (\mathbf{v}_2 \cdot \mathbf{v}_2) \\ &= (\mathbf{v}_2 \cdot \mathbf{w}_3) - 0 - (\mathbf{v}_2 \cdot \mathbf{w}_3) \\ &= 0.\end{aligned}$$

Continuing in this way, we deduce that each of the vectors \mathbf{v}_i is orthogonal to all the previous ones. It follows that $\mathbf{v}_i \cdot \mathbf{v}_j = 0$ for all i, j with $i \neq j$, and hence that $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is an orthogonal basis for \mathbb{R}^n . ■

Exercise C78

Apply the Gram–Schmidt orthogonalisation process to the following basis for \mathbb{R}^5 :

$$\{(1, 0, 0, 0, 0), (0, 2, 0, 0, 0), (0, 0, 1, 1, 0), (1, 1, 1, 1, 1), (1, 0, -1, 0, 1)\}.$$

(You showed, in Exercise C74, that $\{(1, 0, 0, 0, 0), (0, 2, 0, 0, 0), (0, 0, 1, 1, 0)\}$ is an orthogonal set in \mathbb{R}^5 .)

5.4 Orthonormal bases

You have seen that using orthogonal basis vectors can be helpful. However, in many examples it is also useful to require one further condition – that the basis vectors are all unit vectors, as in the standard basis for \mathbb{R}^n .

Recall, from Unit A1, that the magnitude of a vector \mathbf{v} in \mathbb{R}^2 or \mathbb{R}^3 is

$$|\mathbf{v}| = \sqrt{\mathbf{v} \cdot \mathbf{v}}.$$

For example, if $\mathbf{v} = (5, -12)$, then $|\mathbf{v}| = \sqrt{5^2 + (-12)^2} = \sqrt{169} = 13$, as illustrated in Figure 15.

We can similarly define the magnitude of a vector in \mathbb{R}^n , for any positive integer n .

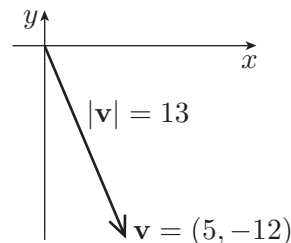


Figure 15 The magnitude of the vector $(5, -12)$

Definition

Let $\mathbf{v} = (v_1, v_2, \dots, v_n)$ be a vector in \mathbb{R}^n . Then the **magnitude** of \mathbf{v} is

$$|\mathbf{v}| = \sqrt{\mathbf{v} \cdot \mathbf{v}} = \sqrt{v_1^2 + v_2^2 + \dots + v_n^2}.$$

Exercise C79

Calculate the magnitude of each of the following vectors.

- (a) $(3, -4, 5)$ in \mathbb{R}^3 . (b) $(1, 2, -1, 0, 3)$ in \mathbb{R}^5 .

Exercise C80

Prove that if \mathbf{v} is any non-zero vector in \mathbb{R}^n , then the vector

$$\frac{\mathbf{v}}{|\mathbf{v}|} = \frac{1}{|\mathbf{v}|} \mathbf{v}$$

has magnitude 1.

We make the following important definition.

Definition

An **orthonormal basis** for \mathbb{R}^n is an orthogonal basis in which each basis vector has magnitude 1.

An orthonormal basis is therefore comprised of orthogonal unit vectors.

It follows from the result of Exercise C80 that, given an orthogonal basis for \mathbb{R}^n , we can obtain an orthonormal basis by scalar multiplication: we need to multiply each basis vector by the reciprocal of its magnitude. This leads to the following strategy for constructing an orthonormal basis.

Strategy C13

To construct an orthonormal basis for \mathbb{R}^n from an orthogonal basis $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ for \mathbb{R}^n :

1. calculate the magnitude of each basis vector
2. scalar multiply each basis vector by the reciprocal of its magnitude.

The required orthonormal basis is $\left\{ \frac{\mathbf{v}_1}{|\mathbf{v}_1|}, \frac{\mathbf{v}_2}{|\mathbf{v}_2|}, \dots, \frac{\mathbf{v}_n}{|\mathbf{v}_n|} \right\}$.

As a shorthand for ‘scalar multiply a vector by the reciprocal of its magnitude’, we may say ‘divide a vector by its magnitude’.

For example, we can use Strategy C13 to obtain an orthonormal basis for \mathbb{R}^3 starting with the orthogonal basis $\{(2, 1, 1), (1, -4, 2), (-2, 1, 3)\}$, as follows. We calculate the magnitude of each basis vector:

$$\begin{aligned} |(2, 1, 1)| &= \sqrt{2^2 + 1^2 + 1^2} = \sqrt{6}, \\ |(1, -4, 2)| &= \sqrt{1^2 + (-4)^2 + 2^2} = \sqrt{21}, \\ |(-2, 1, 3)| &= \sqrt{(-2)^2 + 1^2 + 3^2} = \sqrt{14}. \end{aligned}$$

Dividing each orthogonal basis vector by its magnitude, we obtain the orthonormal basis

$$\left\{ \frac{1}{\sqrt{6}}(2, 1, 1), \frac{1}{\sqrt{21}}(1, -4, 2), \frac{1}{\sqrt{14}}(-2, 1, 3) \right\}.$$

Exercise C81

Construct an orthonormal basis for \mathbb{R}^4 , starting with the basis

$$\{(1, 2, 1, 0), (-1, 1, -1, 1), (1, 0, -1, 0), (1, -1, 1, 3)\}.$$

(You showed, in Exercise C75, that this is an orthogonal basis for \mathbb{R}^4 .)

Note that some of our earlier results become much simpler if we use an orthonormal basis, rather than an orthogonal one. For example, Theorem C34 takes the following form because $\mathbf{v}_i \cdot \mathbf{v}_i = 1$ for each $i \leq n$.

Theorem C36

Let $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ be an orthonormal basis for \mathbb{R}^n , and let \mathbf{u} be any vector in \mathbb{R}^n . Then

$$\mathbf{u} = (\mathbf{v}_1 \cdot \mathbf{u})\mathbf{v}_1 + (\mathbf{v}_2 \cdot \mathbf{u})\mathbf{v}_2 + \cdots + (\mathbf{v}_n \cdot \mathbf{u})\mathbf{v}_n.$$

5.5 Other vector spaces

We conclude this section by remarking that it is possible to define scalar products in vector spaces other than \mathbb{R}^n . For example, in the vector space P_3 we can define the scalar product of two polynomials p_1 and p_2 by

$$p_1 \cdot p_2 = \int_{-1}^1 p_1(x)p_2(x) dx.$$

Such a scalar product is a real number and has properties that are very similar to those of the scalar product in \mathbb{R}^n – for example, $p_1 \cdot p_2 = p_2 \cdot p_1$ for any polynomials p_1 and p_2 .

We can then define such concepts as *orthogonal polynomials*, the *magnitude of a polynomial*, and the *distance and angle between two polynomials*. For example, the polynomials $p_1(x) = x$ and $p_2(x) = x^2$ are orthogonal, since

$$p_1 \cdot p_2 = \int_{-1}^1 x \cdot x^2 dx = \left[\frac{1}{4}x^4 \right]_{-1}^1 = 0$$

and the magnitude of p_2 is given by

$$|p_2|^2 = p_2 \cdot p_2 = \int_{-1}^1 x^2 \cdot x^2 dx = \left[\frac{1}{5}x^5 \right]_{-1}^1 = \frac{2}{5},$$

$$\text{so } |p_2| = \sqrt{\frac{2}{5}}.$$

Although such concepts may seem at first sight to make little sense intuitively, they have proved to be of great interest and importance, for example in mathematical physics. They also show that the mathematical structures we have introduced theoretically here can have surprising applications in other contexts.

Summary

In this unit you have seen how familiar properties of \mathbb{R}^2 and \mathbb{R}^3 can be generalised to other, very different sets of *vectors* through the concept of a *vector space*.

Your study of vector spaces has been driven by looking at properties of \mathbb{R}^2 and \mathbb{R}^3 , such as linear combinations, linear independence and spanning sets of vectors. You have seen how the familiar concept of axes and our intuitive idea of dimension relate to bases of these spaces. You have seen how these concepts generalise to \mathbb{R}^n and other, very different vector spaces such as P_n , $M_{m,n}$ and \mathbb{C} . You have met the Basis Theorem, which states that every basis for a given vector space has the same number of vectors, and that this number is the dimension of the vector space.

Starting with subspaces of \mathbb{R}^2 and \mathbb{R}^3 that can be visualised geometrically, you have seen that subspaces of vector spaces are subsets that are themselves vector spaces, in the same way that subgroups are subsets of groups that are themselves groups.

Finally, you have seen how the scalar product and orthogonality of vectors in \mathbb{R}^n can be used to find orthogonal and orthonormal bases, which are particularly straightforward to work with.

Vector spaces will underpin the remainder of the linear algebra units; in particular you will study functions between vector spaces in Unit C3 *Linear transformations* and use orthonormal bases to classify conics and quadrics in Unit C4 *Eigenvectors*.

Learning outcomes

After working through this unit, you should be able to:

- understand the definition of a *real vector space*
- check whether or not a given set of elements forms a vector space under the operations of vector addition and scalar multiplication
- explain the meaning of the terms *linear combination*, *span* and *spanning set*
- form linear combinations of vectors in a given set
- check whether a vector can be expressed as a linear combination of given vectors
- find the set spanned by a given set of vectors
- check whether a given set of vectors spans the vector space to which the vectors belong
- explain the meaning of the terms *linear independence*, *linear dependence*, *basis* and *dimension*
- test whether a given set of vectors is linearly independent
- test whether a given set of vectors is a basis for a given vector space
- find the E -coordinate representation of a vector given in standard coordinates, and vice versa
- explain what is meant by a *subspace* of a vector space
- test whether a given subset of a vector space is a subspace
- find a basis for a subspace, and hence find its dimension
- check whether the vectors in a given set are *orthogonal*
- express a given vector in terms of an *orthogonal basis*
- use the Gram–Schmidt orthogonalisation process to find orthogonal bases in \mathbb{R}^n
- given an orthogonal basis, construct an orthonormal basis.

Solutions to exercises

Solution to Exercise C44

$$\begin{aligned}\mathbf{u} + \mathbf{v} &= (1, -1, 2, 0, -3) + (0, 2, -1, 4, 0) \\ &= (1, 1, 1, 4, -3) \\ -3\mathbf{u} &= -3(1, -1, 2, 0, -3) \\ &= (-3, 3, -6, 0, 9)\end{aligned}$$

Solution to Exercise C45

Let $\mathbf{u} = (u_1, u_2, u_3, u_4)$, $\mathbf{v} = (v_1, v_2, v_3, v_4)$ and $\mathbf{w} = (w_1, w_2, w_3, w_4)$.

$$\begin{aligned}\text{(a)} \quad (\mathbf{u} + \mathbf{v}) + \mathbf{w} &= ((u_1, u_2, u_3, u_4) + (v_1, v_2, v_3, v_4)) \\ &\quad + (w_1, w_2, w_3, w_4) \\ &= (u_1 + v_1, u_2 + v_2, u_3 + v_3, u_4 + v_4) \\ &\quad + (w_1, w_2, w_3, w_4) \\ &= (u_1 + v_1 + w_1, u_2 + v_2 + w_2, u_3 + v_3 + w_3, \\ &\quad u_4 + v_4 + w_4),\end{aligned}$$

$$\begin{aligned}\mathbf{u} + (\mathbf{v} + \mathbf{w}) &= (u_1, u_2, u_3, u_4) \\ &\quad + ((v_1, v_2, v_3, v_4) + (w_1, w_2, w_3, w_4)) \\ &= (u_1, u_2, u_3, u_4) \\ &\quad + (v_1 + w_1, v_2 + w_2, v_3 + w_3, v_4 + w_4) \\ &= (u_1 + v_1 + w_1, u_2 + v_2 + w_2, u_3 + v_3 + w_3, \\ &\quad u_4 + v_4 + w_4).\end{aligned}$$

Therefore $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$, and so the associative property (A2) holds.

$$\begin{aligned}\text{(b)} \quad \mathbf{v} + (-\mathbf{v}) &= (v_1, v_2, v_3, v_4) + (-v_1, -v_2, -v_3, -v_4) \\ &= (v_1 - v_1, v_2 - v_2, v_3 - v_3, v_4 - v_4) \\ &= (0, 0, 0, 0) = \mathbf{0}\end{aligned}$$

Also, using the commutative property (A5) (proved in Worked Exercise C22(a)) we have

$$\mathbf{v} + (-\mathbf{v}) = \mathbf{0} = -\mathbf{v} + \mathbf{v},$$

so the additive inverses property (A4) holds.

Solution to Exercise C46

$$\begin{aligned}\text{(a)} \quad (p_1(x) + p_2(x)) + p_3(x) &= ((a_1 + b_1x + c_1x^2) + (a_2 + b_2x + c_2x^2)) \\ &\quad + (a_3 + b_3x + c_3x^2) \\ &= ((a_1 + a_2) + (b_1 + b_2)x + (c_1 + c_2)x^2) \\ &\quad + (a_3 + b_3x + c_3x^2) \\ &= (a_1 + a_2 + a_3) + (b_1 + b_2 + b_3)x \\ &\quad + (c_1 + c_2 + c_3)x^2\end{aligned}$$

and

$$\begin{aligned}p_1(x) + (p_2(x) + p_3(x)) &= (a_1 + b_1x + c_1x^2) \\ &\quad + ((a_2 + b_2x + c_2x^2) + (a_3 + b_3x + c_3x^2)) \\ &= (a_1 + b_1x + c_1x^2) \\ &\quad + ((a_2 + a_3) + (b_2 + b_3)x + (c_2 + c_3)x^2) \\ &= (a_1 + a_2 + a_3) + (b_1 + b_2 + b_3)x \\ &\quad + (c_1 + c_2 + c_3)x^2.\end{aligned}$$

Therefore

$$\begin{aligned}(p_1(x) + p_2(x)) + p_3(x) &= p_1(x) + (p_2(x) + p_3(x)),\end{aligned}$$

and so the associative property (A2) holds for addition in P_3 .

(b) We have $\mathbf{0} = 0 + 0x + 0x^2$, so

$$\begin{aligned}p_1(x) + \mathbf{0} &= (a_1 + b_1x + c_1x^2) + (0 + 0x + 0x^2) \\ &= (a_1 + 0) + (b_1 + 0)x + (c_1 + 0)x^2 \\ &= a_1 + b_1x + c_1x^2 = p_1(x)\end{aligned}$$

Also, using the commutative property (A5) (proved in Worked Exercise C23(a)) we have

$$p_1(x) + \mathbf{0} = p_1(x) = \mathbf{0} + p_1(x),$$

so the additive identity property (A3) holds for addition in P_3 .

Solution to Exercise C47

$$\begin{aligned}\text{(a)} \quad 1 \times p(x) &= 1 \times (1 - x + 2x^2) \\ &= 1 \times 1 - 1 \times x + 1 \times 2x^2 \\ &= 1 - x + 2x^2 = p(x),\end{aligned}$$

and therefore the identity property (S3) holds here.

$$\begin{aligned}
 \text{(b)} \quad \alpha(\beta p(x)) &= 2(-3(1-x+2x^2)) \\
 &= 2(-3+3x-6x^2) \\
 &= -6+6x-12x^2 \\
 &= -6(1-x+2x^2) = (\alpha\beta)p(x),
 \end{aligned}$$

and therefore the associative property (S2) holds here.

Solution to Exercise C48

(a) Consider $(1, 3)$ and $(2, 5)$, both in V . Then $(1, 3) + (2, 5) = (3, 8)$, which does not belong to the set V , since $2 \times 3 + 1 = 7 \neq 8$. So the set is not closed under vector addition.

Therefore the set of all ordered pairs (x, y) with $y = 2x + 1$ fails to satisfy the closure axiom (A1), so is not a real vector space.

Alternatively, note that for $(0, 0) \in \mathbb{R}^2$ we have $2 \times 0 + 1 = 1 \neq 0$, so the zero vector is not in V and the additive identity axiom (A3) fails.

Other axioms also fail or do not make sense.

(b) Consider the matrix $\mathbf{A} = \begin{pmatrix} 0 & 2 \\ -1 & 3 \end{pmatrix}$ and

$\alpha = \frac{1}{2}$. Then $\alpha\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -\frac{1}{2} & \frac{3}{2} \end{pmatrix}$, which does not belong to the set.

Therefore the set of matrices of the form

$$\begin{pmatrix} 0 & a \\ b & c \end{pmatrix} \quad \text{with } a, b, c \in \mathbb{Z}$$

fails to satisfy the closure axiom (S1), so is not a real vector space.

Note that axioms A1–A5 and S3 do all hold here, but since axiom S1 fails, the axioms S2, D1 and D2 are meaningless.

Solution to Exercise C49

$$\begin{aligned}
 \text{(a)} \quad 4\mathbf{v}_1 - 2\mathbf{v}_2 &= 4(0, 3) - 2(2, 1) \\
 &= (0, 12) - (4, 2) = (-4, 10)
 \end{aligned}$$

$$\begin{aligned}
 \text{(b)} \quad 3\mathbf{v}_1 + 2\mathbf{v}_2 &= 3(1, 2, 1, 3) + 2(2, 1, 0, -1) \\
 &= (3, 6, 3, 9) + (4, 2, 0, -2) \\
 &= (7, 8, 3, 7)
 \end{aligned}$$

Solution to Exercise C50

$$\begin{aligned}
 \text{(a)} \quad 2\mathbf{v}_1 - 4\mathbf{v}_2 &= 2(2 - x + 3x^2) - 4(-1 + x) \\
 &= (4 - 2x + 6x^2) - (-4 + 4x) \\
 &= 8 - 6x + 6x^2
 \end{aligned}$$

$$\text{(b)} \quad 2\mathbf{v}_1 - 4\mathbf{v}_2 = 2 \sin x - 4x \cos x$$

$$\begin{aligned}
 \text{(c)} \quad 2\mathbf{v}_1 - 4\mathbf{v}_2 &= 2 \begin{pmatrix} -1 & 1 \\ 2 & 0 \end{pmatrix} - 4 \begin{pmatrix} 3 & 1 \\ 0 & -2 \end{pmatrix} \\
 &= \begin{pmatrix} -2 & 2 \\ 4 & 0 \end{pmatrix} - \begin{pmatrix} 12 & 4 \\ 0 & -8 \end{pmatrix} \\
 &= \begin{pmatrix} -14 & -2 \\ 4 & 8 \end{pmatrix}
 \end{aligned}$$

Solution to Exercise C51

We apply Strategy C6.

(a) Let α and β be real numbers such that

$$(2, 4) = \alpha(0, 3) + \beta(2, 1) = (2\beta, 3\alpha + \beta).$$

Equating corresponding coordinates, we obtain the system

$$\begin{aligned}
 2\beta &= 2 \\
 3\alpha + \beta &= 4.
 \end{aligned}$$

The first equation gives $\beta = 1$, and substituting this into the second equation gives $\alpha = 1$, so

$$(2, 4) = (0, 3) + (2, 1).$$

(You might have spotted this linear combination without performing the calculations – it is always worth checking there is not an obvious solution before diving into a strategy!)

(b) Let α , β and γ be real numbers such that

$$\begin{aligned}
 (2, 3, -2) &= \alpha(0, 1, 0) + \beta(1, 2, -1) + \gamma(1, 1, -2) \\
 &= (\beta + \gamma, \alpha + 2\beta + \gamma, -\beta - 2\gamma).
 \end{aligned}$$

Equating corresponding coordinates, we obtain the system

$$\begin{aligned}
 \beta + \gamma &= 2 \\
 \alpha + 2\beta + \gamma &= 3 \\
 -\beta - 2\gamma &= -2.
 \end{aligned}$$

Adding the first and third equations gives $\gamma = 0$, and substituting this into the first equation gives $\beta = 2$. Substituting both these values into the second equation gives $\alpha = -1$, so

$$(2, 3, -2) = -(0, 1, 0) + 2(1, 2, -1) + 0(1, 1, -2).$$

(c) Let α and β be real numbers such that

$$\begin{pmatrix} 3 & 1 \\ 0 & 4 \end{pmatrix} = \alpha \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix} + \beta \begin{pmatrix} 0 & -2 \\ 0 & 1 \end{pmatrix} \\ = \begin{pmatrix} \alpha & -\alpha - 2\beta \\ 0 & 2\alpha + \beta \end{pmatrix}.$$

Equating corresponding entries, we obtain the system

$$\begin{aligned} \alpha &= 3 \\ -\alpha - 2\beta &= 1 \\ 2\alpha + \beta &= 4. \end{aligned}$$

The first equation gives $\alpha = 3$, and substituting this into the second equation gives $\beta = -2$. These values also satisfy the third equation, so

$$\begin{pmatrix} 3 & 1 \\ 0 & 4 \end{pmatrix} = 3 \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix} - 2 \begin{pmatrix} 0 & -2 \\ 0 & 1 \end{pmatrix}.$$

Solution to Exercise C52

(a) We write

$$\begin{aligned} (1, 5, 4) &= \alpha \mathbf{v}_1 + \beta \mathbf{v}_2 \\ &= \alpha(1, 0, 3) + \beta(0, 2, 0) = (\alpha, 2\beta, 3\alpha). \end{aligned}$$

Equating corresponding coordinates, we obtain the system

$$\begin{aligned} \alpha &= 1 \\ 2\beta &= 5 \\ 3\alpha &= 4. \end{aligned}$$

This system is inconsistent and therefore has no solution. So $(1, 5, 4)$ does not lie in the subset of \mathbb{R}^3 spanned by $\{\mathbf{v}_1, \mathbf{v}_2\}$; that is, $(1, 5, 4)$ does not belong to $\langle\{\mathbf{v}_1, \mathbf{v}_2\}\rangle$.

(b) We write

$$\begin{aligned} (1, 5, 4) &= \alpha \mathbf{v}_1 + \beta \mathbf{v}_2 + \gamma \mathbf{v}_3 \\ &= \alpha(1, 0, 3) + \beta(0, 2, 0) + \gamma(0, 3, 1) \\ &= (\alpha, 2\beta + 3\gamma, 3\alpha + \gamma). \end{aligned}$$

Equating corresponding coordinates, we obtain the system

$$\begin{aligned} \alpha &= 1 \\ 2\beta + 3\gamma &= 5 \\ 3\alpha + \gamma &= 4. \end{aligned}$$

The first equation gives $\alpha = 1$, and substituting this into the third gives $\gamma = 1$. Substituting this into the second equation gives $\beta = 1$, so $(1, 5, 4)$ lies in the subset of \mathbb{R}^3 spanned by $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$;

that is, $(1, 5, 4)$ belongs to $\langle\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}\rangle$ and it can be written as

$$(1, 5, 4) = 1(1, 0, 3) + 1(0, 2, 0) + 1(0, 3, 1).$$

(You might have spotted this and avoided following the formal method.)

Solution to Exercise C53

(a) Each vector in \mathbb{R}^2 can be written as (x, y) . To show that (x, y) is in $\langle\{(1, 1), (-1, 2)\}\rangle$, we write

$$\begin{aligned} (x, y) &= \alpha(1, 1) + \beta(-1, 2) \\ &= (\alpha - \beta, \alpha + 2\beta). \end{aligned}$$

Equating corresponding coordinates, we obtain the system

$$\begin{aligned} \alpha - \beta &= x \\ \alpha + 2\beta &= y. \end{aligned}$$

These equations have solution $\alpha = \frac{1}{3}(2x + y)$ and $\beta = \frac{1}{3}(y - x)$, so any vector in \mathbb{R}^2 can be written in terms of $(1, 1)$ and $(-1, 2)$ as

$$(x, y) = \frac{1}{3}(2x + y)(1, 1) + \frac{1}{3}(y - x)(-1, 2).$$

So $\{(1, 1), (-1, 2)\}$ is a spanning set for \mathbb{R}^2 .

(b) Each vector in \mathbb{R}^2 can be written as (x, y) . To show that (x, y) is in $\langle\{(2, -1), (3, 2)\}\rangle$, we write

$$\begin{aligned} (x, y) &= \alpha(2, -1) + \beta(3, 2) \\ &= (2\alpha + 3\beta, -\alpha + 2\beta). \end{aligned}$$

Equating corresponding coordinates, we obtain the system

$$\begin{aligned} 2\alpha + 3\beta &= x \\ -\alpha + 2\beta &= y. \end{aligned}$$

These equations have solution $\alpha = \frac{1}{7}(2x - 3y)$ and $\beta = \frac{1}{7}(x + 2y)$, so any vector in \mathbb{R}^2 can be written in terms of $(2, -1)$ and $(3, 2)$ as

$$(x, y) = \frac{1}{7}(2x - 3y)(2, -1) + \frac{1}{7}(x + 2y)(3, 2).$$

So $\{(2, -1), (3, 2)\}$ is a spanning set for \mathbb{R}^2 .

Solution to Exercise C54

We write

$$\begin{aligned} (x, y, z) &= \alpha(1, 0, 0) + \beta(1, 1, 0) + \gamma(2, 0, 1) \\ &= (\alpha + \beta + 2\gamma, \beta, \gamma). \end{aligned}$$

Equating corresponding coordinates, we obtain the system

$$\begin{aligned}\alpha + \beta + 2\gamma &= x \\ \beta &= y \\ \gamma &= z.\end{aligned}$$

Working backwards from the third equation, we find that these equations have solution $\gamma = z$, $\beta = y$ and $\alpha = x - y - 2z$, so any vector in \mathbb{R}^3 can be written in terms of $(1, 0, 0)$, $(1, 1, 0)$ and $(2, 0, 1)$ as

$$\begin{aligned}(x, y, z) &= (x - y - 2z)(1, 0, 0) \\ &\quad + y(1, 1, 0) + z(2, 0, 1).\end{aligned}$$

So $\{(1, 0, 0), (1, 1, 0), (2, 0, 1)\}$ is a spanning set for \mathbb{R}^3 .

Solution to Exercise C55

Each polynomial in P_4 can be written as $a + bx + cx^2 + dx^3$. To show that $a + bx + cx^2 + dx^3$ belongs to $\langle\{1 + x, 1 + x^2, 1 + x^3, x\}\rangle$, we write

$$\begin{aligned}a + bx + cx^2 + dx^3 &= \alpha(1 + x) + \beta(1 + x^2) + \gamma(1 + x^3) + \delta x \\ &= (\alpha + \beta + \gamma) + (\alpha + \delta)x + \beta x^2 + \gamma x^3.\end{aligned}$$

Equating corresponding coefficients, we obtain the system

$$\begin{aligned}\alpha + \beta + \gamma &= a \\ \alpha &+ \delta = b \\ \beta &= c \\ \gamma &= d.\end{aligned}$$

It has solution $\gamma = d$, $\beta = c$, $\alpha = a - c - d$ and $\delta = b - a + c + d$. So

$$\begin{aligned}a + bx + cx^2 + dx^3 &= (a - c - d)(1 + x) + c(1 + x^2) + d(1 + x^3) \\ &\quad + (b - a + c + d)x.\end{aligned}$$

Thus $\langle\{1 + x, 1 + x^2, 1 + x^3, x\}\rangle = P_4$.

Solution to Exercise C56

(a) We have

$$\begin{aligned}\langle S \rangle &= \{\alpha(1, 0, 0) : \alpha \in \mathbb{R}\} \\ &= \{(\alpha, 0, 0) : \alpha \in \mathbb{R}\}.\end{aligned}$$

(Geometrically, $\langle S \rangle$ is the x -axis.)

(b) We have

$$\begin{aligned}\langle S \rangle &= \left\{ \alpha \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} + \beta \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix} : \alpha, \beta \in \mathbb{R} \right\} \\ &= \left\{ \begin{pmatrix} 2\alpha - \beta & 0 \\ 0 & 3\alpha + 2\beta \end{pmatrix} : \alpha, \beta \in \mathbb{R} \right\}.\end{aligned}$$

Thus

$$\langle S \rangle \subseteq \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} : a, b \in \mathbb{R} \right\}.$$

To show that every 2×2 diagonal matrix belongs to $\langle S \rangle$, we write

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} 2\alpha - \beta & 0 \\ 0 & 3\alpha + 2\beta \end{pmatrix}.$$

Equating corresponding entries, we obtain the system

$$\begin{aligned}2\alpha - \beta &= a \\ 3\alpha + 2\beta &= b.\end{aligned}$$

It has solution

$$\begin{aligned}\alpha &= \frac{1}{7}(2a + b) \\ \beta &= \frac{1}{7}(-3a + 2b),\end{aligned}$$

so

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in \langle S \rangle.$$

Hence

$$\langle S \rangle = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} : a, b \in \mathbb{R} \right\}.$$

Solution to Exercise C57

(a) These two vectors are linearly independent because neither is a multiple of the other. (In this case there is no need to use Strategy C7.)

(b) Using Strategy C7, we write

$$\alpha(1, -1) + \beta(1, 1) + \gamma(2, 1) = (0, 0).$$

This gives the system

$$\begin{aligned}\alpha + \beta + 2\gamma &= 0 \\ -\alpha + \beta + \gamma &= 0.\end{aligned}$$

Adding the equations gives $2\beta + 3\gamma = 0$, or $\beta = -\frac{3}{2}\gamma$, and substituting this into the first equation gives $\alpha = -\frac{1}{2}\gamma$; that is, $\gamma = -2\alpha$ and $\beta = 3\alpha$. The solution set of the system is

$$\alpha = k, \beta = 3k, \gamma = -2k, \quad k \in \mathbb{R},$$

so there are infinitely many solutions. For example, $k = 1$ gives

$$(1, -1) + 3(1, 1) - 2(2, 1) = (0, 0).$$

So the set $\{(1, -1), (1, 1), (2, 1)\}$ is linearly dependent.

Alternatively, you may have expressed the solution set here in terms of γ and found another solution – *any* solution (where α , β and γ are not all zero) is sufficient to show that the vectors are linearly dependent.

(c) These two vectors are linearly independent because neither is a multiple of the other. (In this case there is no need to use Strategy C7.)

(d) We write

$$\alpha(1, 0, 0) + \beta(1, 1, 0) + \gamma(1, 1, 1) = (0, 0, 0).$$

This gives the system

$$\begin{aligned}\alpha + \beta + \gamma &= 0 \\ \beta + \gamma &= 0 \\ \gamma &= 0.\end{aligned}$$

The third equation gives $\gamma = 0$, and substituting into the second equation gives $\beta = 0$. Finally, substituting into the first equation gives $\alpha = 0$. The only solution is $\alpha = \beta = \gamma = 0$.

Therefore the set $\{(1, 0, 0), (1, 1, 0), (1, 1, 1)\}$ is linearly independent.

(e) These two vectors are linearly independent because neither is a multiple of the other. (Again, there is no need to use Strategy C7.)

Solution to Exercise C58

(a) The set $\{1, x, x^2, x^3, 1 + x + x^2 + x^3\}$ is linearly dependent because the fifth vector is the sum of the first four vectors. So

$$1 + x + x^2 + x^3 - (1 + x + x^2 + x^3) = 0.$$

(b) The set S is linearly independent because neither matrix is a multiple of the other.

(c) We apply Strategy C7.

We write

$$\alpha \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} + \beta \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} + \gamma \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

which can be written as

$$\begin{pmatrix} \alpha + \beta + \gamma & \alpha + \gamma \\ \beta + \gamma & \alpha + \beta + \gamma \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Equating corresponding entries, we obtain the system

$$\begin{aligned}\alpha + \beta + \gamma &= 0 \\ \alpha &+ \gamma = 0 \\ \beta + \gamma &= 0 \\ \alpha + \beta + \gamma &= 0.\end{aligned}$$

Subtracting the second equation from the first, and the third from the fourth, we get $\beta = 0$ and $\alpha = 0$. Substituting these values in the first and fourth gives $\gamma = 0$ also. Therefore the only solution to this system is $\alpha = \beta = \gamma = 0$. Therefore the set S is a linearly independent subset of $M_{2,2}$.

(d) The set $\{1 + i, 1 - i\}$ is linearly independent because neither vector is a (real) multiple of the other.

Solution to Exercise C59

(a) None of the vectors in this set has a non-zero x -component; so whenever $x \neq 0$, we cannot write (x, y, z) in terms of these three vectors.

Therefore this set of vectors is not a basis for \mathbb{R}^3 because it does not span \mathbb{R}^3 .

(If you had not spotted the zero x -component and had followed Strategy C8, you would have discovered that this set is not linearly independent: for example,

$$16(0, 1, 2) - 11(0, 2, 3) + (0, 6, 1) = (0, 0, 0).$$

Therefore this set of vectors is not a basis for \mathbb{R}^3 .)

(b) We check both conditions in Strategy C8.

Using Strategy C7, we write

$$\alpha(1, 2, 1) + \beta(1, 0, -1) + \gamma(0, 3, 1) = (0, 0, 0),$$

which simplifies to

$$(\alpha + \beta, 2\alpha + 3\gamma, \alpha - \beta + \gamma) = (0, 0, 0).$$

Equating corresponding coordinates, we obtain the system

$$\begin{aligned}\alpha + \beta &= 0 \\ 2\alpha &+ 3\gamma = 0 \\ \alpha - \beta &+ \gamma = 0.\end{aligned}$$

Adding the third equation to the first gives $2\alpha + \gamma = 0$, and subtracting this from the second equation gives $\gamma = 0$. Substituting this into the second equation gives $\alpha = 0$. Finally, substituting $\alpha = 0$ into the first equation gives $\beta = 0$. The only solution is $\alpha = \beta = \gamma = 0$.

Therefore the set is linearly independent.

We apply Strategy C6.

Each vector in \mathbb{R}^3 can be written as (x, y, z) , with $x, y, z \in \mathbb{R}$. To show that (x, y, z) is in

$$\langle \{(1, 2, 1), (1, 0, -1), (0, 3, 1)\} \rangle,$$

we write

$$(x, y, z) = \alpha(1, 2, 1) + \beta(1, 0, -1) + \gamma(0, 3, 1).$$

Equating corresponding coordinates, we obtain the system

$$\begin{aligned} \alpha + \beta &= x \\ 2\alpha &+ 3\gamma = y \\ \alpha - \beta + \gamma &= z. \end{aligned}$$

Adding the third equation to the first gives $2\alpha + \gamma = x + z$, and subtracting this from the second equation gives $\gamma = \frac{1}{2}(y - x - z)$.

Substituting this into the second equation gives $\alpha = \frac{1}{4}(3x - y + 3z)$. Finally, substituting for α in the first equation gives $\beta = \frac{1}{4}(x + y - 3z)$. We have a solution, so any vector in \mathbb{R}^3 can be written as

$$\begin{aligned} (x, y, z) &= \frac{1}{4}(3x - y + 3z)(1, 2, 1) \\ &+ \frac{1}{4}(x + y - 3z)(1, 0, -1) \\ &+ \frac{1}{2}(y - x - z)(0, 3, 1). \end{aligned}$$

Therefore the set of vectors spans \mathbb{R}^3 .

Thus $\{(1, 2, 1), (1, 0, -1), (0, 3, 1)\}$ is a basis for \mathbb{R}^3 .

(c) Here we have

$$(1, 1, 1) = (1, 0, 0) + (0, 1, 0) + (0, 0, 1),$$

so these vectors are not linearly independent.

Therefore the set

$$\{(1, 0, 0), (0, 1, 0), (0, 0, 1), (1, 1, 1)\}$$

is not a basis for \mathbb{R}^3 .

Solution to Exercise C60

We check both conditions in Strategy C8.

This set is linearly independent because there are only two vectors in the set, and neither vector is a multiple of the other.

We apply Strategy C6.

Each vector in \mathbb{R}^4 can be written as (x, y, z, w) , with $x, y, z, w \in \mathbb{R}$. To show that (x, y, z, w) is in

$$\langle \{(1, 2, -1, -1), (-1, 5, 1, 3)\} \rangle,$$

we write

$$(x, y, z, w) = \alpha(1, 2, -1, -1) + \beta(-1, 5, 1, 3).$$

Equating corresponding coordinates, we obtain the system

$$\begin{aligned} \alpha - \beta &= x \\ 2\alpha + 5\beta &= y \\ -\alpha + \beta &= z \\ -\alpha + 3\beta &= w. \end{aligned}$$

Adding the first and third equations gives $x + z = 0$. This contradicts the assumption that x, y, z and w can take any real values, so

$$\{(1, 2, -1, -1), (-1, 5, 1, 3)\}$$

is not a spanning set for \mathbb{R}^4 .

Thus the set $\{(1, 2, -1, -1), (-1, 5, 1, 3)\}$ is not a basis for \mathbb{R}^4 .

Solution to Exercise C61

We check both conditions in Strategy C8.

Using Strategy C7 we write

$$\begin{aligned} \alpha \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} + \beta \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + \gamma \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} + \delta \begin{pmatrix} -3 & 1 \\ 0 & 0 \end{pmatrix} \\ = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{aligned}$$

which simplifies to

$$\begin{pmatrix} \alpha + 2\gamma - 3\delta & -\beta + \delta \\ \alpha + \beta & \gamma \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Equating corresponding entries, we obtain the system

$$\begin{aligned} \alpha &+ 2\gamma - 3\delta = 0 \\ -\beta &+ \delta = 0 \\ \alpha + \beta &= 0 \\ \gamma &= 0. \end{aligned}$$

From the fourth equation we have $\gamma = 0$, and from the second and third $\alpha = -\beta = -\delta$. Substituting into the first equation gives $\alpha + 3\alpha = 0$, so $\alpha = 0$. The only solution is therefore $\alpha = \beta = \gamma = \delta = 0$.

Therefore the set is linearly independent.

We apply Strategy C6.

Each 2×2 matrix can be written as $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, with $a, b, c, d \in \mathbb{R}$. To show this is in $\langle S \rangle$ we write

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \alpha \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} + \beta \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \\ + \gamma \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} + \delta \begin{pmatrix} -3 & 1 \\ 0 & 0 \end{pmatrix}.$$

Equating corresponding entries, we obtain the system

$$\begin{array}{rcl} \alpha & + 2\gamma - 3\delta & = a \\ & -\beta & + \delta = b \\ \alpha + \beta & & = c \\ & \gamma & = d. \end{array}$$

From the fourth equation we have $\gamma = d$, and adding the second equation to the third gives $\alpha + \delta = b + c$. Substituting for γ in the first equation gives $\alpha - 3\delta = a - 2d$. These last two equations give $\delta = \frac{1}{4}(b + c - a + 2d)$.

Then, by substitution, $\alpha = \frac{1}{4}(a + 3b + 3c - 2d)$ and $\beta = \frac{1}{4}(-a - 3b + c + 2d)$.

We have a solution $\alpha = \frac{1}{4}(a + 3b + 3c - 2d)$,
 $\beta = \frac{1}{4}(-a - 3b + c + 2d)$, $\gamma = d$ and
 $\delta = \frac{1}{4}(b + c - a + 2d)$.

Therefore the set of matrices S spans the set $M_{2,2}$ of all 2×2 matrices.

Thus S is a basis for $M_{2,2}$.

Solution to Exercise C62

(a) For the basis $E = \{(1, 2), (-3, 1)\}$, we have

$$\begin{aligned}(2, 1)_E &= 2(1, 2) + 1(-3, 1) \\ &= (2, 4) + (-3, 1) \\ &= (-1, 5).\end{aligned}$$

(b) For the basis

$E = \{(1, 0, 2), (-1, 1, 3), (2, -2, 0)\}$, we have

$$\begin{aligned}(1, 1, -1)_E &= 1(1, 0, 2) + 1(-1, 1, 3) - 1(2, -2, 0) \\ &= (1, 0, 2) + (-1, 1, 3) - (2, -2, 0) \\ &= (-2, 3, 5).\end{aligned}$$

Solution to Exercise C63

(a) We write

$$(5, -4) = \alpha(1, 2) + \beta(-3, 1).$$

Equating corresponding coordinates, we obtain the system

$$\begin{array}{rcl} \alpha - 3\beta & = & 5 \\ 2\alpha + \beta & = & -4. \end{array}$$

Solving these equations gives $\alpha = -1$, $\beta = -2$, so

$$\begin{aligned}(5, -4) &= -1(1, 2) - 2(-3, 1) \\ &= (-1, -2)_E.\end{aligned}$$

(b) We write

$$(-3, 5, 7) = \alpha(1, 0, 2) + \beta(-1, 1, 3) + \gamma(2, -2, 0).$$

Equating corresponding coordinates, we obtain the system

$$\begin{array}{rcl} \alpha - \beta + 2\gamma & = & -3 \\ \beta - 2\gamma & = & 5 \\ 2\alpha + 3\beta & = & 7. \end{array}$$

Adding the first and second equations gives $\alpha = 2$, and substituting this into the third equation gives $\beta = 1$. Substituting for β in the second equation gives $\gamma = -2$. So

$$\begin{aligned} (-3, 5, 7) &= 2(1, 0, 2) + 1(-1, 1, 3) - 2(2, -2, 0) \\ &= (2, 1, -2)_E. \end{aligned}$$

Solution to Exercise C64

We apply Strategy C9.

(a) This set contains only two vectors, not three, so cannot be a basis for \mathbb{R}^3 .

(Neither vector is a multiple of the other, so it is however linearly independent.)

(b) This set contains three vectors, so it may be a basis for \mathbb{R}^3 .

We write

$$\alpha(1, 0, 1) + \beta(1, 0, -1) + \gamma(0, 1, 1) = (0, 0, 0).$$

Equating corresponding coordinates, we obtain the system

$$\begin{aligned}\alpha + \beta &= 0 \\ \gamma &= 0 \\ \alpha - \beta + \gamma &= 0.\end{aligned}$$

The second equation gives $\gamma = 0$. Substituting this into the third equation gives $\alpha - \beta = 0$. Adding this new equation to the first equation gives $\alpha = 0$ and hence $\beta = 0$. The only solution is $\alpha = \beta = \gamma = 0$.

Therefore the set is linearly independent.

The set contains three vectors and is linearly independent; therefore it is a basis for \mathbb{R}^3 .

(c) Here we have

$$(1, -1, 0) + (2, 1, 4) = (3, 0, 4),$$

so this set is not linearly independent.

Therefore this set is not a basis for \mathbb{R}^3 .

(It does however contain the correct number of vectors.)

(d) This set contains four vectors, so it cannot be a basis for \mathbb{R}^3 .

(Alternatively, here we have

$$(1, 1, 1) = (1, 0, 0) + (0, 1, 0) + (0, 0, 1),$$

so this set is also linearly dependent.)

Solution to Exercise C65

We apply Strategy C9.

(a) This set contains four vectors and $M_{2,2}$ has dimension 4, so it may be a basis.

Using Strategy C7 we write

$$\begin{aligned}\alpha \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} + \beta \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} + \gamma \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} + \delta \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \\ = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\end{aligned}$$

which simplifies to

$$\begin{pmatrix} \alpha + \gamma & \beta + \gamma + \delta \\ \alpha + \delta & \beta + \delta \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Equating corresponding entries, we obtain the system

$$\begin{aligned}\alpha + \gamma &= 0 \\ \beta + \gamma + \delta &= 0 \\ \alpha + \delta &= 0 \\ \beta + \delta &= 0.\end{aligned}$$

From the first, third and fourth equations we have $\alpha = \beta = -\gamma = -\delta$. Substituting in the second gives $-\beta = 0$. The only solution is therefore $\alpha = \beta = \gamma = \delta = 0$.

Therefore the set is linearly independent.

The set S contains four vectors and is linearly independent so is a basis for $M_{2,2}$.

(Compare the length of this solution to that of Exercise C61 using Strategy C8.)

(b) This set contains two vectors and P_2 has dimension 2, so it may be a basis.

This set is linearly independent because there are only two vectors in the set, and neither vector is a multiple of the other.

So by Strategy C9, the set is a basis for P_2 .

Solution to Exercise C66

The set S is a subset of \mathbb{R}^2 , so we use Strategy C10.

If $x = 0$, then $(x, -2x) = (0, 0)$, so S contains the zero vector of \mathbb{R}^2 .

Let $\mathbf{v}_1 = (x_1, -2x_1)$ and $\mathbf{v}_2 = (x_2, -2x_2)$ belong to S . Then

$$\begin{aligned}\mathbf{v}_1 + \mathbf{v}_2 &= (x_1, -2x_1) + (x_2, -2x_2) \\ &= (x_1 + x_2, -2x_1 - 2x_2) \\ &= (x_1 + x_2, -2(x_1 + x_2)).\end{aligned}$$

This vector has the correct form for a vector in S , since $x_1 + x_2 \in \mathbb{R}$, so S is closed under vector addition.

Let $\mathbf{v} = (x, -2x) \in S$ and $\alpha \in \mathbb{R}$. Then

$$\begin{aligned}\alpha \mathbf{v} &= \alpha(x, -2x) \\ &= (\alpha x, \alpha(-2x)) \\ &= (\alpha x, -2(\alpha x)).\end{aligned}$$

This vector has the correct form for a vector in S , since $\alpha x \in \mathbb{R}$, so S is closed under scalar multiplication.

Since conditions (1), (2) and (3) are satisfied, S is a subspace of \mathbb{R}^2 .

(This subspace is the line through the origin with equation $y = -2x$.)

Solution to Exercise C67

In each case the set S is a subset of V , so we use Strategy C10.

(a) If $\mathbf{0} \in S$, then $(x, x+2) = (0, 0)$ for some number x . Equating coordinates, we obtain the system

$$\begin{aligned}x &= 0 \\x &= -2.\end{aligned}$$

This system is inconsistent so has no solution. Therefore $\mathbf{0}$ does not belong to S and condition (1) is not satisfied. Hence S is not a subspace of \mathbb{R}^2 .

(b) If $x = y = z = 0$, then

$$(x, y, z, x+2y-z) = (0, 0, 0, 0),$$

so S contains the zero vector of \mathbb{R}^4 .

Let $\mathbf{v}_1 = (x_1, y_1, z_1, x_1 + 2y_1 - z_1)$ and $\mathbf{v}_2 = (x_2, y_2, z_2, x_2 + 2y_2 - z_2)$ belong to S . Then

$$\begin{aligned}\mathbf{v}_1 + \mathbf{v}_2 &= (x_1, y_1, z_1, x_1 + 2y_1 - z_1) \\&\quad + (x_2, y_2, z_2, x_2 + 2y_2 - z_2) \\&= (x_1 + x_2, y_1 + y_2, z_1 + z_2, \\&\quad x_1 + 2y_1 - z_1 + x_2 + 2y_2 - z_2) \\&= (x_1 + x_2, y_1 + y_2, z_1 + z_2, \\&\quad (x_1 + x_2) + 2(y_1 + y_2) - (z_1 + z_2)).\end{aligned}$$

This vector has the correct form for a vector in S , since $x_1 + x_2, y_1 + y_2, z_1 + z_2 \in \mathbb{R}$, so S is closed under vector addition.

Let $\mathbf{v} = (x, y, z, x + 2y - z) \in S$ and $\alpha \in \mathbb{R}$. Then

$$\begin{aligned}\alpha\mathbf{v} &= \alpha(x, y, z, x + 2y - z) \\&= (\alpha x, \alpha y, \alpha z, \alpha(x + 2y - z)) \\&= (\alpha x, \alpha y, \alpha z, (\alpha x) + 2(\alpha y) - (\alpha z)).\end{aligned}$$

This vector has the correct form for a vector in S , since $\alpha x, \alpha y, \alpha z \in \mathbb{R}$, so S is closed under scalar multiplication.

Since conditions (1), (2) and (3) are satisfied, S is a subspace of \mathbb{R}^4 .

Solution to Exercise C68

In each case the set S is a subset of V , so we use Strategy C10.

(a) The zero vector of P_3 is $0 + 0x + 0x^2 = \mathbf{0}$. If $a = b = 0$, then $p(x) = 0 + 0x = \mathbf{0}$, so S contains the zero vector.

Let $p_1(x) = a_1 + b_1x$ and $p_2(x) = a_2 + b_2x$ belong to S . Then

$$\begin{aligned}p_1(x) + p_2(x) &= a_1 + b_1x + a_2 + b_2x \\&= (a_1 + a_2) + (b_1 + b_2)x.\end{aligned}$$

This polynomial has the correct form for a vector in S , since $a_1 + a_2, b_1 + b_2 \in \mathbb{R}$, so S is closed under vector addition.

Let $p(x) = a + bx \in S$ and $\alpha \in \mathbb{R}$. Then

$$\alpha p(x) = \alpha a + \alpha bx = (\alpha a) + (\alpha b)x.$$

This polynomial has the correct form for a vector in S , since $\alpha a, \alpha b \in \mathbb{R}$, so S is closed under scalar multiplication.

Since conditions (1), (2) and (3) are satisfied, S is a subspace of V .

(b) The zero vector of P_3 is $0 + 0x + 0x^2 = \mathbf{0}$, which is not of the form $x + ax^2$ for a vector in S . Therefore $\mathbf{0}$ does not belong to S and condition (1) fails. Hence S is not a subspace of P_3 .

(Alternatively, you may have spotted that conditions (2) and (3) also fail. Using a particularly simple vector can make the calculations to show this easy: by setting for example $a = 0$, we see that $p(x) = x$ belongs to S . The sum $p(x) + p(x) = 2x$, however, does not belong to S , and for $\alpha \in \mathbb{R}$ not equal to 1, the scalar product αx is also not in S .)

(c) The zero vector of $M_{2,2}$ is $\mathbf{0} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, which

is not of the form $\begin{pmatrix} a & 1 \\ 0 & d \end{pmatrix}$ for a vector in S .

Therefore $\mathbf{0}$ does not belong to S and condition (1) fails. Hence S is not a subspace of $M_{2,2}$.

Solution to Exercise C69

Since $\{(1, -2, 0), (0, 3, 3)\}$ is a linearly independent set, the subspace it spans is a two-dimensional subspace of \mathbb{R}^3 and is therefore a plane through the origin with equation

$$ax + by + cz = 0,$$

where a, b, c are not all zero.

Since the vectors in the spanning set lie in the plane, the values of a, b and c must satisfy the system

$$\begin{aligned} a - 2b &= 0 \\ 3b + 3c &= 0. \end{aligned}$$

The first of these equations gives $a = 2b$, and the second equation gives $c = -b$, so the subspace is the plane with equation $2bx + by - bz = 0$, or, equivalently,

$$2x + y - z = 0.$$

Solution to Exercise C70

Since

$$\begin{aligned} (x, y, z, x + 2y - z) &= (x, 0, 0, x) + (0, y, 0, 2y) + (0, 0, z, -z) \\ &= x(1, 0, 0, 1) + y(0, 1, 0, 2) + z(0, 0, 1, -1), \end{aligned}$$

any vector in S can be written as a linear combination of the vectors in the set

$$\{(1, 0, 0, 1), (0, 1, 0, 2), (0, 0, 1, -1)\},$$

so this set spans S .

To check whether these vectors are linearly independent, we write

$$\begin{aligned} \alpha(1, 0, 0, 1) + \beta(0, 1, 0, 2) + \gamma(0, 0, 1, -1) \\ = (0, 0, 0, 0). \end{aligned}$$

This gives the system

$$\begin{aligned} \alpha &= 0 \\ \beta &= 0 \\ \gamma &= 0 \\ \alpha + 2\beta - \gamma &= 0, \end{aligned}$$

and hence $\alpha = \beta = \gamma = 0$. Therefore the set is linearly independent.

So $\{(1, 0, 0, 1), (0, 1, 0, 2), (0, 0, 1, -1)\}$ is a basis for S . Therefore S has dimension 3.

Solution to Exercise C71

$$\begin{aligned} \text{(a)} \quad (2, 1, 1) \cdot (1, -4, 2) &= 2 \times 1 + 1 \times (-4) + 1 \times 2 \\ &= 2 - 4 + 2 = 0, \end{aligned}$$

so $(2, 1, 1)$ and $(1, -4, 2)$ are orthogonal.

$$\text{(b)} \quad \mathbf{v}_1 \cdot \mathbf{v}_2 = -2 \times 9 + 6 \times 2 + 1 \times 6 = 0,$$

so \mathbf{v}_1 and \mathbf{v}_2 are orthogonal.

$$\begin{aligned} \mathbf{v}_1 \cdot \mathbf{v}_3 &= -2 \times 4 + 6 \times (-15) + 1 \times (-1) \\ &= -99, \end{aligned}$$

which is non-zero, so \mathbf{v}_1 and \mathbf{v}_3 are not orthogonal.

$$\begin{aligned} \mathbf{v}_2 \cdot \mathbf{v}_3 &= 9 \times 4 + 2 \times (-15) \\ &\quad + 6 \times (-1) = 0, \end{aligned}$$

so \mathbf{v}_2 and \mathbf{v}_3 are orthogonal.

Solution to Exercise C72

(a) Let $\mathbf{v}_1 = (3, 4, 0)$, $\mathbf{v}_2 = (8, -6, 0)$ and $\mathbf{v}_3 = (0, 0, 5)$. Then

$$\begin{aligned} \mathbf{v}_1 \cdot \mathbf{v}_2 &= 3 \times 8 + 4 \times (-6) + 0 \times 0 = 0, \\ \mathbf{v}_1 \cdot \mathbf{v}_3 &= 3 \times 0 + 4 \times 0 + 0 \times 5 = 0, \\ \mathbf{v}_2 \cdot \mathbf{v}_3 &= 8 \times 0 + (-6) \times 0 + 0 \times 5 = 0. \end{aligned}$$

Thus $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is an orthogonal set in \mathbb{R}^3 . Since there are three non-zero vectors in this set, it is an orthogonal basis for \mathbb{R}^3 .

(b) We apply Strategy C11.

$$\begin{aligned} \alpha_1 &= \frac{\mathbf{v}_1 \cdot \mathbf{u}}{\mathbf{v}_1 \cdot \mathbf{v}_1} \\ &= \frac{(3, 4, 0) \cdot (10, 0, 4)}{(3, 4, 0) \cdot (3, 4, 0)} \\ &= \frac{30}{25} = \frac{6}{5}, \\ \alpha_2 &= \frac{\mathbf{v}_2 \cdot \mathbf{u}}{\mathbf{v}_2 \cdot \mathbf{v}_2} \\ &= \frac{(8, -6, 0) \cdot (10, 0, 4)}{(8, -6, 0) \cdot (8, -6, 0)} \\ &= \frac{80}{100} = \frac{4}{5}, \end{aligned}$$

and

$$\begin{aligned} \alpha_3 &= \frac{\mathbf{v}_3 \cdot \mathbf{u}}{\mathbf{v}_3 \cdot \mathbf{v}_3} \\ &= \frac{(0, 0, 5) \cdot (10, 0, 4)}{(0, 0, 5) \cdot (0, 0, 5)} \\ &= \frac{20}{25} = \frac{4}{5}. \end{aligned}$$

Thus $(10, 0, 4) = \frac{6}{5}(3, 4, 0) + \frac{4}{5}(8, -6, 0) + \frac{4}{5}(0, 0, 5)$.

Solution to Exercise C73

- (a) $(1, 2, -1, 0) \cdot (0, -5, 6, -3)$
 $= 1 \times 0 + 2 \times (-5) + (-1) \times 6 + 0 \times (-3)$
 $= 0 - 10 - 6 + 0$
 $= -16$
- (b) $(1, 2, 3, 4, 5, 6) \cdot (3, 2, 1, 0, -1, -2)$
 $= 1 \times 3 + 2 \times 2 + 3 \times 1 + 4 \times 0$
 $+ 5 \times (-1) + 6 \times (-2)$
 $= 3 + 4 + 3 + 0 - 5 - 12$
 $= -7$

Solution to Exercise C74

We check that each pair of vectors is orthogonal by forming the scalar product of each pair of vectors in the set:

$$\begin{aligned}(1, 0, 0, 0, 0) \cdot (0, 2, 0, 0, 0) &= 0 + 0 + 0 + 0 + 0 \\ &= 0, \\ (1, 0, 0, 0, 0) \cdot (0, 0, 1, 1, 0) &= 0 + 0 + 0 + 0 + 0 \\ &= 0, \\ (0, 2, 0, 0, 0) \cdot (0, 0, 1, 1, 0) &= 0 + 0 + 0 + 0 + 0 \\ &= 0.\end{aligned}$$

Therefore these three vectors form an orthogonal set in \mathbb{R}^5 .

Solution to Exercise C75

We check that each pair of vectors is orthogonal by forming the scalar product of each pair of vectors in the set:

$$\begin{aligned}(1, 2, 1, 0) \cdot (-1, 1, -1, 1) &= -1 + 2 - 1 + 0 \\ &= 0, \\ (1, 2, 1, 0) \cdot (1, 0, -1, 0) &= 1 + 0 - 1 + 0 \\ &= 0, \\ (1, 2, 1, 0) \cdot (1, -1, 1, 3) &= 1 - 2 + 1 + 0 \\ &= 0, \\ (-1, 1, -1, 1) \cdot (1, 0, -1, 0) &= -1 + 0 + 1 + 0 \\ &= 0, \\ (-1, 1, -1, 1) \cdot (1, -1, 1, 3) &= -1 - 1 - 1 + 3 \\ &= 0, \\ (1, 0, -1, 0) \cdot (1, -1, 1, 3) &= 1 + 0 - 1 + 0 \\ &= 0.\end{aligned}$$

Therefore these vectors form an orthogonal set in \mathbb{R}^4 . Since there are four, non-zero vectors in this set, these vectors form an orthogonal basis for \mathbb{R}^4 by Corollary C33.

Solution to Exercise C76

We apply Strategy C12.

Let $\mathbf{v}_1 = (1, 2, 1, 0)$, $\mathbf{v}_2 = (-1, 1, -1, 1)$, $\mathbf{v}_3 = (1, 0, -1, 0)$, $\mathbf{v}_4 = (1, -1, 1, 3)$ and $\mathbf{u} = (1, 2, 3, 4)$. Then

$$\begin{aligned}\alpha_1 &= \frac{\mathbf{v}_1 \cdot \mathbf{u}}{\mathbf{v}_1 \cdot \mathbf{v}_1} = \frac{8}{6} = \frac{4}{3}, \\ \alpha_2 &= \frac{\mathbf{v}_2 \cdot \mathbf{u}}{\mathbf{v}_2 \cdot \mathbf{v}_2} = \frac{2}{4} = \frac{1}{2}, \\ \alpha_3 &= \frac{\mathbf{v}_3 \cdot \mathbf{u}}{\mathbf{v}_3 \cdot \mathbf{v}_3} = \frac{-2}{2} = -1, \\ \alpha_4 &= \frac{\mathbf{v}_4 \cdot \mathbf{u}}{\mathbf{v}_4 \cdot \mathbf{v}_4} = \frac{14}{12} = \frac{7}{6}.\end{aligned}$$

Thus

$$\begin{aligned}(1, 2, 3, 4) &= \frac{4}{3}(1, 2, 1, 0) + \frac{1}{2}(-1, 1, -1, 1) \\ &\quad - (1, 0, -1, 0) + \frac{7}{6}(1, -1, 1, 3).\end{aligned}$$

Solution to Exercise C77

(a) Using $\mathbf{x} \cdot \mathbf{n} = 0$, we have

$$(x, y, z) \cdot (3, -4, 5) = 0;$$

that is, the equation of the plane is

$$3x - 4y + 5z = 0.$$

(b) We have

$$\begin{aligned}\mathbf{w}_1 \cdot \mathbf{n} &= (4, 3, 0) \cdot (3, -4, 5) \\ &= 12 - 12 + 0 = 0,\end{aligned}$$

and

$$\begin{aligned}\mathbf{w}_2 \cdot \mathbf{n} &= (0, 5, 4) \cdot (3, -4, 5) \\ &= 0 - 20 + 20 = 0,\end{aligned}$$

so both these vectors lie in the plane.

(Alternatively, rather than using the vector equation of the plane, we can check that the points $(4, 3, 0)$ and $(0, 5, 4)$ satisfy the equation $3x - 4y + 5z = 0$ of the plane.)

(c) We set $\mathbf{v}_1 = (4, 3, 0)$ and

$$\begin{aligned}\mathbf{v}_2 &= \mathbf{w}_2 - \frac{\mathbf{v}_1 \cdot \mathbf{w}_2}{\mathbf{v}_1 \cdot \mathbf{v}_1} \mathbf{v}_1 \\ &= (0, 5, 4) - \frac{(4, 3, 0) \cdot (0, 5, 4)}{(4, 3, 0) \cdot (4, 3, 0)} (4, 3, 0) \\ &= (0, 5, 4) - \frac{15}{25} (4, 3, 0) \\ &= (0, 5, 4) - \frac{3}{5} (4, 3, 0) \\ &= \left(-\frac{12}{5}, \frac{16}{5}, 4\right).\end{aligned}$$

The required orthogonal basis for the plane is

$$\left\{(4, 3, 0), \left(-\frac{12}{5}, \frac{16}{5}, 4\right)\right\}.$$

(d) An orthogonal basis for \mathbb{R}^3 is

$$\left\{(3, -4, 5), (4, 3, 0), \left(-\frac{12}{5}, \frac{16}{5}, 4\right)\right\}.$$

Solution to Exercise C78

We apply Theorem C35 with $\mathbf{w}_1 = (1, 0, 0, 0, 0)$, $\mathbf{w}_2 = (0, 2, 0, 0, 0)$, $\mathbf{w}_3 = (0, 0, 1, 1, 0)$, $\mathbf{w}_4 = (1, 1, 1, 1, 1)$ and $\mathbf{w}_5 = (1, 0, -1, 0, 1)$.

Since \mathbf{w}_1 , \mathbf{w}_2 and \mathbf{w}_3 already form an orthogonal set, we have

$$\begin{aligned}\mathbf{v}_1 &= \mathbf{w}_1 = (1, 0, 0, 0, 0), \\ \mathbf{v}_2 &= \mathbf{w}_2 = (0, 2, 0, 0, 0), \\ \mathbf{v}_3 &= \mathbf{w}_3 = (0, 0, 1, 1, 0).\end{aligned}$$

Then

$$\begin{aligned}\mathbf{v}_4 &= \mathbf{w}_4 - \left(\frac{\mathbf{v}_1 \cdot \mathbf{w}_4}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right) \mathbf{v}_1 - \left(\frac{\mathbf{v}_2 \cdot \mathbf{w}_4}{\mathbf{v}_2 \cdot \mathbf{v}_2}\right) \mathbf{v}_2 \\ &\quad - \left(\frac{\mathbf{v}_3 \cdot \mathbf{w}_4}{\mathbf{v}_3 \cdot \mathbf{v}_3}\right) \mathbf{v}_3 \\ &= (1, 1, 1, 1, 1) - \frac{1}{1}(1, 0, 0, 0, 0) \\ &\quad - \frac{2}{4}(0, 2, 0, 0, 0) - \frac{2}{2}(0, 0, 1, 1, 0) \\ &= (0, 0, 0, 0, 1)\end{aligned}$$

and

$$\begin{aligned}\mathbf{v}_5 &= \mathbf{w}_5 - \left(\frac{\mathbf{v}_1 \cdot \mathbf{w}_5}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right) \mathbf{v}_1 - \left(\frac{\mathbf{v}_2 \cdot \mathbf{w}_5}{\mathbf{v}_2 \cdot \mathbf{v}_2}\right) \mathbf{v}_2 \\ &\quad - \left(\frac{\mathbf{v}_3 \cdot \mathbf{w}_5}{\mathbf{v}_3 \cdot \mathbf{v}_3}\right) \mathbf{v}_3 - \left(\frac{\mathbf{v}_4 \cdot \mathbf{w}_5}{\mathbf{v}_4 \cdot \mathbf{v}_4}\right) \mathbf{v}_4 \\ &= (1, 0, -1, 0, 1) - \frac{1}{1}(1, 0, 0, 0, 0) \\ &\quad - 0 - \left(-\frac{1}{2}\right)(0, 0, 1, 1, 0) - \frac{1}{1}(0, 0, 0, 0, 1) \\ &= \left(0, 0, -\frac{1}{2}, \frac{1}{2}, 0\right).\end{aligned}$$

Thus we have the orthogonal basis

$$\left\{(1, 0, 0, 0, 0), (0, 2, 0, 0, 0), (0, 0, 1, 1, 0), \left(0, 0, 0, 0, 1\right), \left(0, 0, -\frac{1}{2}, \frac{1}{2}, 0\right)\right\}.$$

Solution to Exercise C79

$$\begin{aligned}\text{(a)} \quad (3, -4, 5) \cdot (3, -4, 5) &= 9 + 16 + 25 \\ &= 50,\end{aligned}$$

$$\text{so } |(3, -4, 5)| = \sqrt{50} = 5\sqrt{2}.$$

$$\begin{aligned}\text{(b)} \quad (1, 2, -1, 0, 3) \cdot (1, 2, -1, 0, 3) &= 1 + 4 + 1 + 0 + 9 \\ &= 15,\end{aligned}$$

$$\text{so } |(1, 2, -1, 0, 3)| = \sqrt{15}.$$

Solution to Exercise C80

If $\mathbf{v} = (v_1, v_2, \dots, v_n)$ is a non-zero vector, then

$$\frac{\mathbf{v}}{|\mathbf{v}|} = \left(\frac{v_1}{\sqrt{\mathbf{v} \cdot \mathbf{v}}}, \frac{v_2}{\sqrt{\mathbf{v} \cdot \mathbf{v}}}, \dots, \frac{v_n}{\sqrt{\mathbf{v} \cdot \mathbf{v}}}\right),$$

so the magnitude of $\mathbf{v}/|\mathbf{v}|$ is

$$\begin{aligned}&\sqrt{\frac{v_1^2}{\mathbf{v} \cdot \mathbf{v}} + \frac{v_2^2}{\mathbf{v} \cdot \mathbf{v}} + \dots + \frac{v_n^2}{\mathbf{v} \cdot \mathbf{v}}} \\ &= \sqrt{\frac{v_1^2 + v_2^2 + \dots + v_n^2}{\mathbf{v} \cdot \mathbf{v}}} \\ &= \sqrt{\frac{\mathbf{v} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}}} = 1.\end{aligned}$$

Solution to Exercise C81

We apply Strategy C13.

We have

$$\begin{aligned}|(1, 2, 1, 0)| &= \sqrt{6}, \\ |(-1, 1, -1, 1)| &= \sqrt{4} = 2, \\ |(1, 0, -1, 0)| &= \sqrt{2}, \\ |(1, -1, 1, 3)| &= \sqrt{12} = 2\sqrt{3}.\end{aligned}$$

The required orthonormal basis for \mathbb{R}^4 is therefore

$$\left\{\frac{1}{\sqrt{6}}(1, 2, 1, 0), \frac{1}{2}(-1, 1, -1, 1), \frac{1}{\sqrt{2}}(1, 0, -1, 0), \frac{1}{2\sqrt{3}}(1, -1, 1, 3)\right\}.$$